Harmonic sums and polylogarithms at non-positive integers

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We are interested in the implementation in Maple of the following objects

\[ H_{-s_1, \ldots, -s_r}(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} n_1^{s_1} \ldots n_r^{s_r}, \forall N \in \mathbb{N}_+, \] (1)

\[ \text{Li}_{-s_1, \ldots, -s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} z^{n_1^{s_1} \ldots n_r^{s_r}}, \forall z \in \mathbb{C}, \] (2)

\[ \zeta(-s_1, \ldots, -s_k) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} (1 + n_1)^{s_1} \ldots (k + n_1 + \ldots + n_k)^{s_k}, \] (3)

where \( s_1, \ldots, s_k \) are non-negative integers. 1.

Precisely,
- We prove that \( H_{-s_1, \ldots, -s_r}(N) \) is a polynomial of degree \( s_1 + \ldots + s_r + r \) of \( N \) and we give the explicit formula to compute the constants \( C_{-s_1, \ldots, -s_r} \) such that
  \[ \lim_{N \to \infty} \frac{C_{-s_1, \ldots, -s_r} N^{s_1 + \ldots + s_r + r}}{H_{-s_1, \ldots, -s_r}(N)} = 1. \] (4)

- We prove also that \( \text{Li}_{-s_1, \ldots, -s_r}(z) \) is a polynomial of degree \( s_1 + \ldots + s_r + r \) of \( (1 - z)^{-1} \) and we give the explicit formula to compute the constants \( B_{s_1, \ldots, s_r} \) such that
  \[ \lim_{z \to 1^{-}} \frac{B_{s_1, \ldots, s_r} (1 - z)^{-(s_1 + \ldots + s_r + r)}}{\text{Li}_{s_1, \ldots, s_r}(z)} = 1. \] (5)

1. Quantities in eq. 3 are divergent, the aim of this work is to give tools to approach these divergences.
- We give the shuffle structure for (1).
- We study the values of (3) at negative integers, by analytic prolongation.

Bibliographie


