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THE SET OF LOGARITHMICALLY CONVERGENT SEQUENCES CANNOT BE ACCELERATED*

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Abstract. Some theorems (Pennacchi, Germain-Bonne, Smith and Ford) state that methods of a certain form which are exact on geometric sequences accelerate linear convergence. But no corresponding theorem is known for logarithmic convergence. Our study shows the reason why: There is no algorithm which can accelerate all logarithmically convergent sequences. We obtain this result with a generalization of "remanence", which is a sufficient property for a set of sequences to be unaccelerable.

Introduction. In a recent paper [10] D. A. Smith and W. F. Ford have studied the acceleration of convergence of sequences, and in particular the acceleration of logarithmically convergent sequences. In the conclusions of their article [10 p. 238], they say there is empirical evidence that for sequence transformations a good test of applicability to logarithmic convergence is exactness on series of Cordellier's types [3]. They ask if one can rigorously establish such a result (analogous to "Germain-Bonne's theorem" [7] concerning linear convergence). We propose the following answer:

Such a result cannot exist because there is no algorithm which accelerates convergence of every logarithmically convergent sequence.

The reasons are similar to those of [6], but it is necessary to define a generalization of the notion of *remanence* (introduced in [6]), which is a sufficient property for a set of sequences to be unaccelerable.

We begin with some definitions and remarks about convergence and acceleration of convergence in § 1.

In § 2 we give the generalization of remanence and the corresponding "negative theorem": A set of sequences which has generalized remanence is not accelerable. In § 3 we establish that the set of logarithmically convergent sequences has generalized remanence.

The conclusions are that logarithmic convergence is intrinsically difficult to accelerate (unlike linear convergence) and that it is necessary to use various methods according to the kind of logarithmic convergence. (Levin's u transform [9] seems the best available across-the-board method, but for certain subclasses of logarithmically convergent sequences, ε, ρ and θ-algorithms are to be preferred [1], [2].)

1. Logarithmic convergence, algorithms for sequences and acceleration.

(a) Let (x_n) be a real convergent sequence with limit x. We say that (x_n) is *logarithmically convergent* if and only if:

$$\lim_{n\to\infty} \frac{x_{n+1}-x}{x_n-x}=1, \tag{11}$$

$$\lim_{n\to\infty} \frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}=1. \tag{12}$$

We denote by L the set of all logarithmically convergent sequences.

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If we delete (12) we obtain a more general definition of logarithmic convergence and a corresponding set of sequences denoted by $\mathscr{L}$. In [6] we have shown that $\mathscr{L}$ cannot be accelerated. This first result was easy to obtain compared with the result about L established in § 3.

(b) The notion of "algorithm for sequences" used in this work is a very general one. It contains, for example, every algorithm of the form

$$
\begin{aligned}
t_0 &= f_0(x_0, x_1, \cdots, x_k)\\
t_1 &= f_1(x_0, x_1, \cdots, x_{k+1})\\
&\cdots \quad \cdots \quad \cdots\\
t_n &= f_n(x_0, x_1, \cdots, x_{k+n})\\
&\cdots \quad \cdots \quad \cdots
\end{aligned}
$$

where $f_0, f_1, \cdots, f_n, \cdots$ are functions of $k+1,\ k+2,\ \cdots,\ k+n+1, \cdots$ variables, respectively. For more details and precise definitions see [4], [5], [6].

(c) We say that the sequence (t_n) accelerates the convergent sequence (x_n) if and only if $\lim_{n\to\infty}(t_n - x)/(x_n - x) = 0$.

2. Generalized remanence. Let $\mathscr{S}$ be a set of real convergent sequences; we say that $\mathscr{S}$ possesses the property of *generalized remanence* if and only if

GR

a) There exists a convergent sequence $(\hat{x}_n)$ with limit $\hat{x}$ satisfying $\forall n \in \mathbb{N}$: $\hat{x}_n \neq \hat{x}$, and such that:
   1) there exists $(x_n^0) \in \mathscr{S}$ such that $(x_n^0) \to \hat{x}_0$,
   2) for every $m_0 \geqq 0$, there exist $p_0 \geqq m_0$ and $(x_n^1) \in \mathscr{S}$ such that $(x_n^1) \to \hat{x}_1$ and $\forall m \leqq p_0$: $x_m^1 = x_m^0$,
   3) for every $m_1 > p_0$, there exist $p_1 \geqq m_1$ and $(x_n^2) \in \mathscr{S}$ such that $(x_n^2) \to \hat{x}_2$ and $\forall m \leqq p_1$: $x_m^2 = x_m^1$
   $\cdots \quad \cdots \quad \cdots$.

b) $(x_0^0, x_1^0, \cdots, x_{p_0}^0, x_{p_0+1}^1, x_{p_0+2}^1, \cdots, x_{p_1}^2, x_{p_1+1}^2, \cdots, x_{p_2}^2, x_{p_2+1}^3, \cdots) \in \mathscr{S}$.

The word "remanence" has been chosen for the following reason: Starting from a sequence (x_n^0) belonging to $\mathscr{S}$, it is possible to construct a set of sequences (x_x^p) in $\mathscr{S}(p = 1, 2, \cdots)$ and a new sequence (construction GR b) which still belongs to $\mathscr{S}$; hence, the property of belonging to $\mathscr{S}$ persists (*remains*).

In [6], we defined, for a set $\mathscr{S}$, the property of remanence:

R

a) There exists a convergent sequence $(\hat{x}_n)$ with limit $\hat{x}$ satisfying: $\forall n \in \mathbb{N}$: $\hat{x}_n \neq \hat{x}$, and such that:
   1) there exists $(x_n^0) \in \mathscr{S}$ such that $(x_n^0) \to \hat{x}_0$,
   2) for every $m_0 \in \mathbb{N}$, there exists $(x_n^1) \in \mathscr{S}$ such that $(x_n^1) \to \hat{x}_1$ and $\forall n \leqq m_0$: $x_n^1 = x_n^0$,
   3) for every $m_1 \geqq m_0$, there exists $(x_n^2) \in \mathscr{S}$ such that $(x_n^2) \to \hat{x}_2$ and $\forall n \leqq m_1$: $x_n^2 = x_n^1$
   $\cdots \quad \cdots \quad \cdots$.

b) $(x_0^0, x_1^0, \cdots, x_{m_0}^0, x_{m_0+1}^1, \cdots, x_{m_1}^1, x_{m_1+1}^2, \cdots, x_{m_2}^2, x_{m_2+1}^3 \cdots) \in \mathscr{S}$.

If R is satisfied then GR is satisfied (thus GR is more easily true) and reasoning analogous to that in [6] gives the following "negative result":

PROPOSITION. *If the set of sequences $\mathcal{S}$ satisfies* GR *then there is no algorithm which accelerates the convergence of every sequence of $\mathcal{S}$.*

3. *L* is not accelerable.

THEOREM. *The set L has generalized remanence* GR, *and thus there is no algorithm which accelerates every sequence of L.*

We need the following lemma:

LEMMA. *Let $p_0 \in \mathbb{N}$, $a, b, c, r \in \mathbb{R}$ such that $a < b < c$, $0 < r < 1$, $(c-b)/(1-r) < (c-a)$.*

There exists a sequence (x_n) such that:

(i) $x_{p_0} = c$, $x_{p_0+1} = b$,

(ii) $(x_n) \to a$,

(iii) $\forall n \geqq p_0$: $r \leqq (x_{n+2} - x_{n+1})/(x_{n+1} - x_n) \leqq 1$,

(iv) $[(x_{n+2} - x_{n+1})/(x_{n+1} - x_n)] \to 1$,

(v) $\forall n \geqq p_0$: $r \leqq (x_{n+1} - a)/(x_n - a) \leqq 1$,

(vi) $[(x_{n+1} - a)/(x_n - a)] \to 1$.

Proof of the lemma. Clearly it is sufficient to prove the lemma with $p_0 = 0$. Therefore, we assume that $p_0 = 0$.

Let (r_n) be a real, increasing sequence which converges to 1 and such that $r_0 = r$.

We shall construct convergent sequences $(x_n^0), (x_n^1), \cdots, (x_n^i), \cdots$ (with limits x^0, $x^1, \cdots, x^i, \cdots$ respectively), and then we shall define the desired sequence (x_n).

Construction of (x_n^0).

We put $s_0 = r$, $x_0^0 = c$ and for every $n \geqq 1$

$$x_n^0 = c - (c-b)(1 + s_0 + s_0^2 + \cdots + s_0^{n-1}).$$

We have

$$x^0 = c - \frac{c-b}{1-s_0} = c - \frac{c-b}{1-r} > c - (c-a) = a.$$

Let $m_{-1} = 0$.

Construction of (x_n^{i+1}).

There exist an integer $m_i \geqq m_{i-1}$ and a real s_{i+1} such that

$$(\mathrm{a}_{i+1}) \qquad r_{i+1} < s_{i+1} < 1,$$

$$(\mathrm{b}_{i+1}) \qquad |x_{m_i}^i - x^i| \leqq \frac{1}{2^i},$$

$$(\mathrm{c}_{i+1}) \qquad x_{m_i-1}^i - \frac{x_{m_i-1}^i - x_{m_i}^i}{1 - s_{i+1}} = \frac{x^i + a}{2}.$$

(First, we determine m_i such that

$$|x_{m_i}^i - x^i| \leqq \frac{1}{2^i}, \qquad x_{m_i-1}^i - \frac{x_{m_i-1}^i - x_{m_i}^i}{1 - r_{i+1}} \geqq \frac{x^i + a}{2}.$$

Then we choose s_{i+1}.)

The sequence (x_n^{i+1}) is defined by

$$x_n^{i+1} = \begin{cases} x_n^i & \text{for } n \in \{0, 1, \cdots, m_i\}, \\ x_{m_i-1}^i - (x_{m_i-1}^i - x_{m_i}^i)(1 + s_{i+1} + s_{i+1}^2 + \cdots + s_{i+1}^{n-m_i}) & \text{for } n > m_i. \end{cases}$$

Then we have

$$x^{i+1}=x^i_{m_i-1}-\frac{x^i_{m_i-1}-x^i_{m_i}}{1-s_{i+1}}=\frac{x^i+a}{2}>a.$$

When all the sequences (x^0_n), $(x^1_n), \cdots, (x^i_n), \cdots$ are defined, we put

$$(x_n)=(x^0_0, x^0_1, \cdots, x^0_{m_0}, x^1_{m_0+1}, x^1_{m_0+2}, \cdots, x^1_{m_1}, x^2_{m_1+1}, \cdots).$$

Now we shall prove that for this sequence (i), (ii), (iii), (iv), (v), (vi) are true.

(i) By construction $x^0_0=c$, $x^0_1=b$. Since $p_0=0$, (i) is true.

(ii) The sequence (x_n) decreases. Also, from (b_1), $(b_2), \cdots, (b_i), \cdots$, we obtain

$$\forall i \geqq 1, \quad |x^i - x_{m_i}| \leqq \frac{1}{2^i},$$

while by construction we obtain

$$\forall i \geqq 1, \quad x^i=\frac{x^{i-1}+a}{2}.$$

Therefore (ii) is true.

(iii) and (iv) If $m\in\{m_i-1, m_i, \cdots, m_{i+1}-2\}$, then $(x_{m+2}-x_{m+1})/(x_{m+1}-x_m)=s_{i+1}$. From (a_1), $(a_2), \cdots, (a_i), \cdots$ we obtain $\forall i \in \mathbb{N}$, $r_i \leqq s_i \leqq 1$. Consequently (iii) and (iv) are true.

(v) and (vi) If $m\in\{m_i, m_i-1, \cdots, m_{i+1}-1\}$, then $(x_{m+1}-x^{i+1})/(x_m-x^{i+1})=s_{i+1}$. Since $a<x^{i+1}<x_{m+1}<x_m$, we obtain $s_{i+1}\leqq(x_{m+1}-a)/(x_m-a)\leqq 1$, and thus (v) and (vi) are true.

Proof of the theorem.

(a) We take $\hat{x}_n=1/(n+1)$. Let (v_n) be an increasing sequence of positive real numbers which converges to 1. We define (x^0_n) by putting: $x^0_n=1+1/(n+1)$.

Step 0. Let m_0 be an arbitrary integer. There exists $p_0>m_0$ such that

$$\text{(A1)} \qquad \frac{x^0_{p_0}-x^0_{p_0+1}}{x^0_{p_0}-\hat{x}_1}\leqq 1-v_0,$$

$$\text{(B1)} \qquad |x^0_{p_0}-\hat{x}_0|\leqq\frac{1}{2^0}.$$

Let (y^1_n) be the sequence given by the lemma with p_0, $a=\hat{x}_1$, $b=x^0_{p_0+1}$, $c=x^0_{p_0}$, $r=v_0$. (This sequence exists because (A1) is true.)

The sequence (x^1_n) is defined by putting

$$x^1_n=\begin{cases} x^0_n & \text{for } n\in\{0, 1, \cdots, p_0\}, \\ y^1_n & \text{for } n>p_0. \end{cases}$$

Step i. Let m_i be an arbitrary integer such that $m_i>p_{i-1}$. There exists $p_i>m_i$ satisfying:

$$(\mathrm{A}^{i+1}) \qquad \frac{x^i_{p_i}-x^i_{p_i+1}}{x^i_{p_i}-\hat{x}_{i+1}}\leqq 1-v_i,$$

$$(\mathrm{B}^{i+1}) \qquad |x^i_{p_i}-\hat{x}_i|\leqq\frac{1}{2^i}.$$

Let (y_n^{i+1}) be the sequence given by the lemma with p_i, $a = \hat{x}_{i+1}$, $b = x^i_{p_i+1}$, $c = x^i_{p_i}$, $r = v_i$.

The sequence (x_n^{i+1}) is defined by putting

$$x_n^{i+1} = \begin{cases} x_n^i & \text{for } n \in \{0, 1, \cdots, p_i\}, \\ y_n^{i+1} & \text{for } n > p_i. \end{cases}$$

b) Now we must prove that the sequence $(x_n) = (x^0_0, x^0_1, \cdots, x^0_{p_0}, x^1_{p_0+1}, \cdots, x^1_{p_1}, x^2_{p_1+1}, \cdots)$ is a sequence of L.

From (B1), (B2), $\cdots$, (Bi, $\cdots$, we obtain that (x_n) converges to 0. From (A1), (A2), $\cdots$ (Ai), $\cdots$ and the properties (i) and (iii) of the lemma we have (12).

The proof of (11) is similar to the proof of (v), (vi) in the lemma.

4. Conclusions. Our work leads us to the following new point of view on acceleration of logarithmic convergence: The set of all logarithmically convergent sequences is too large. Hence, if we want to accelerate the convergence of certain logarithmically convergent sequences, we must restrict ourselves to subsets of L.

What are the accelerable subsets of L? This is now the good question. Subsequent studies will determine those subsets, as large as possible but not too large! (At the present time, C. Kowalewski is working in this direction [8].)

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