CHAPTER 8

RANDOMNESS, UNPREDICTABILITY AND ABSENCE OF ORDER: THE IDENTIFICATION BY THE THEORY OF RECURSIVITY OF THE MATHEMATICAL NOTION OF RANDOM SEQUENCE

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1. ABSTRACT

The theory of recursivity which was initiated by Gödel, Church, Turing and Post between 1930 and 1936 leads 30 years later to an absolute definition of randomness that seems to fulfil the main objectives stated by von Mises. The definition of random sequences by Martin-Löf in 1965 and the other works on the so-called 'algorithmic theory of information' by Kolmogorof, Chaitin, Schnorr and Levin (among others) may be understood as the formulation of a thesis similar to the Church-Turing's Thesis about the notion of algorithmic calculability. Here is this new thesis we call the **Martin-Löf-Chaitin's Thesis**: the intuitive informal concept of random sequences (of 0 and 1) is satisfactorily defined by the notion of Martin-Löf-Chaitin random sequences (MLC-random sequences) that is, sequences which do not belong to any recursively null set. In this paper (a short version of [Delahaye 1990]), we first recall and explain shortly the notion of MLC-random sequences; and propose afterwards a comparison between the Church-Turing's Thesis and the Martin-Löf-Chatin's Thesis. Our conclusion is that there is a huge similarity between the two thesis, but that today the Martin-Löf-Chaitin's Thesis is more problematic and more complex than the Church-Turing's Thesis.

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2. INTRODUCTION

In the context of the foundation of probability, the notion of random sequences was introduced by Richard von Mises ([von Mises 1919], [von Mises 1941], [von Mises 1964]) under the name of "collectives" ("Kollectiv"). Von Mises' idea was that a mathematical theory of probability should be based on a precise and absolute definition of randomness. The crucial features of collectives are the existence of limiting relative frequencies within the sequence, and the invariance of the limiting relative frequencies under the operation of 'admissible selection'.

> We claim that our theory, which serves to describe observable facts, satisfies all reasonable requirements of logical consistency and is free from contradictions and obscurities of any kind. ... I would even claim that the real meaning of the Bernoulli theorem is inaccessible to any probability theory that does not start with the frequency definition of probability. ... All axioms of Kolmogorof can be accepted within the framework of our theory as a part of it, but in no way as a substitute for the foregoing definition of probability. *R. von Mises. On the Foundations of Probability and Statistics. Ann. Math. Statis. 12. 1941. pp. 191-205.*

But unfortunately von Mises did not really arrive at a satisfactory notion of 'admissible selection' and consequentially did not give a satisfactory mathematical definition of what he calls "collective," that is random sequences.

> The problem of giving an adequate mathematical definition of a random sequence was subjected to an intense discussion about thirty years ago. It was initiated by von Mises as early as 1919 and reached its climax in the thirties when it engaged most of the pioneers of probability theory of that time. ... Von Mises urged that a mathematical theory of probability should be based on a definition of randomness, the probability of an event then being introduced as the limit of the relative frequency as the number of trials tends to infinity. ... It was objected that there is just as little need for a

> definition of random sequences and probabilities by means of them as there is need for a definition of points and straight lines in geometry. ... The question was not whether the theory in spe should be axiomatized or not, but what objects should be taken as primitive and what axioms should be chosen to govern them. In the axiomatization of Kolmogorof 1933 the random sequences are left outside the theory. ... [Von Mises] wanted to define random sequences in an absolute sense, sequences that were to possess all conceivable properties of stochasticity. This program appears impossible to carry out within the measure theoretic framework of Kolmogorof 1933. ... It seems as if it were this incapability of finding an adequate mathematical definition that brought the so rapid development in the thirties to an abrupt end. ... A common feature of the experiments considered by von Mises is that they may be repeated any, or at least a very large number of times. For the sequence of the successive outcomes $x_1, x_2, \ldots, x_n, \ldots$ which is imagined to extend indefinitely, von Mises coined the term "Kollektiv". A Kollektiv has to satisfy two requirements. To formulate the first of these let n_A denote the frequency with which the event A has occurred in the first n trials, i.e. the number of points x_m $1 \leq m \leq n$, that belong to the subset A of the sample space. For every "angebbare Punktmenge" A the limit of the relative frequencies should exist, lim $n_A/n = pi(A)$. This limit is called the probability of the event A with respect to the given Kollektiv. ... The second axiom is more intricate. It is to express the well-known irregularity of a random sequence, the impossibility of characterizing the correspondence between the number of an experiment and its outcome by a mathematical law. In a gambler's terminology it may be called the axiom of the impossibility of a successful gambling system. Thus sequences like (0 1 0 1 0 1 0 1 0 1 0 1 0 ...), 0 denoting failure and 1 success, are excluded although the limit frequency exists, since betting at every even trial would assure us constant success. The final form of the axiom is the following. If we select a subsequence of $x_1, x_2, \ldots, x_n, \ldots$ in such a way that the decision whether x_n should be selected or not does not depend on x_n, then the limiting relative frequency of the subsequence should exist and equal that of the original sequence. ... The definition of a Kollektiv was criticised for being mathematically imprecise or even inconsistent. ... The trouble was due to the fact that the concept of effectiveness was not a rigorous mathematical one at that time. *P.*

Martin-Löf. The definition of Random Sequences. Information and Control. 9. 1966. pages 602-619.

The axiomatic construction of probability theory on the basis of mesure theory [Kolmogorof 1936] as a purely mathematical discipline is logically irreproachable and does not cast doubts in anybody's mind. However to be able to apply this theory rigorously in practice its **physical interpretation has to be stated clearly**. Until recently there was no satisfactory solution of this problem. Indeed, probability is usually interpreted by means of the following arguments: "If we perform many tests, then the ratio of the number of favourable outcomes to the number of tests performed will always give a number close to, and in the limit exactly equal to, the probability (or measure) of the event in question. However to say "always" here would be untrue: strictly speaking, this does not always happen, but only with probability 1 (and for finite series of tests, with probability close to 1). In this way the concept of the probability of an arbitrary event is defined through the concept of an event that has probability close to (and in the limit equal to) 1, consequently cannot be defined in this manner without an obviously circular argument. In 1919 von Mises put forward the following way of eliminating these difficulties: according to von Mises there are random and non-random sequences. From the mathematical point of view, random sequences form a set of full mesure and all without exception satisfy all the laws of probability theory. It is physically possible to assume that as a result of an experiment only random sequences appear. However, the definition of random sequences proposed by von Mises and later defined more precisely by Wald [1937], Church [1940] and Kolmogorof [1963] turned out to be unsatisfactory. For example, the existence was proved of random sequences, according to von Mises (his so-called collectives) that do not satisfy the law of iterated logarithm [Ville 1939]. *A. K. Zvonkin, L. A. Levin. The Complexity of finite object and the development of the concepts of information and randomness by means of the theory of algorithms. Russ Math. Survey, 25, 6, 1970. pages 83-124.*

3. THE NOTION OF MARTIN-LÖF-CHAITIN RANDOM SEQUENCE

In 1965 Kolmogorof [Kolmogorof 1965] has defined the complexity H(Y) of an object Y as the minimal length of the binary program which computes Y on a certain Universal Turing Machine. He shows that this notion was invariant in the sense that if U and U' are two Universal Machines then the complexity defined by the first is the same as the complexity defined by the second within a constant. Similar work was done simultaneously by Chaitin [Chaitin 1966 1969a]. Kolmogorof on the basis of this definition has proposed to consider those elements of a given large finite population to be random whose complexity is maximal.

In 1966 Martin-Löf had shown that the random element as defined by Kolmogorof possess all conceivable statistical properties of randomness. He also extended the definition to infinite binary sequences and for the first time gave a precise mathematical definition of the von Mises' Kolectivs.

Several equivalent formulations are possible. We give them here. In the following we identify a real number in the interval [0, 1] with its sequence of digits (hence instead of defining the notion of random infinite sequence of 0 and 1, we define the notion of random real).

The 4 following definitions are equivalent: ([Chaitin 1987b])

The real number r in the interval [0, 1] *is* random *if and only if*:

Definition 1 (random in the **Martin-Löf** sense, 1966)

> *For every recursively enumerable sequence A_i of sets of intervals, every A_i with a mesure less than 2^{-i} ($\mu(A) < 2^{-i}$): r does not belong to every A_i.*

Definition 2 (random in the **Solovay** sense, 1975)

> *For every recursively enumerable sequence A_i of sets of intervals with a finite total mesure ($\Sigma\mu(A_i) < \infty$): r is at most a finite number of A_i.*

Definition 3 (random in the **Chaitin-Levin** sense)

The complexity $H(r_n)$ *of the* n *first digits of* r *satisfies:*

$$\exists c\ \forall n:\ H(r_n) \geq n-c$$

Definition 4 (strongly random in the **Chaitin** sense, 1987)

The complexity $H(r_n)$ *of the* n *first digits of* r *satisfies:*

$$\forall k \exists N \forall n \geq N:\ H(r_n) \geq n+k$$

In a less formal manner, the situation of the question today is that:

For a sequence of binary digit, **to be random** is to verify one of the equivalent properties:

- *not to fulfil any exceptional regularity effectively testable* (i.e. to pass all the sequential effective random test [Martin-Löf] 1966]),

- *to have an incompressible information content* (i.e. to have a maximal algorithmic complexity [Levin 1974] [Chaitin 1975a]),

- *to be impredictable or impossible to win* (no gambling system can win when playing on the sequence) [Schnorr 1971a] [Schnorr 1977]).

and **to be random** implies:

- *not to have any algorithmic form* (not to be definissable with an algorithm as the sequence of the digits of Π is)

- *to have limiting relative frequencies for every subsequence extracted by an algorithm* (effective property of von Mises-Church);

- *to be free from aftereffect* [Popper 1935] (weak form of the property of von Mises-Church: the limiting relative frequencies do not change for sequences extracted by the following process: a finite sequence of 0 and 1 being fixed x_1, x_2, ..., x_n, extract the subsequence of elements which are just after each occurrence of x_1, x_2, ..., x_n.)

For a more precise history and mathematical details the reader is referred to [Martin-Löf 1969b] [Zvonkin Levin 1970] [Schnorr 1977] [van Lambalgen 1987] [Kolmogorof Uspenskii 1987] [Li Vitanyi 1989c] [Delahaye 1990] [Li Vitanyi 1990].

4. COMPARISON OF CHURCH-TURING'S THESIS AND MARTIN-LÖF-CHAITIN'S THESIS.

It seems interesting to compare the present situation of the Church-Turing's Thesis (about the notion of "algorithmically calculable function") and the Martin-Löf-Chaitin's Thesis (about the notion of "infinite binary random sequence").

It is necessary to precise that we only want to consider the "standard Church-Turing's Thesis" which identifies the mathematical concept of *recursive function* with the intuitive metamathematical concept of *function calculable with a discrete, deterministic, finite algorithm*. Here we do not consider the physical Church-Turing's Thesis (about functions calculable by machines or physical processes) nor the mental Church-Turing's Thesis (about functions calculable by minds or brains).

The discussion about the Church-Turing's Thesis is often obscured by the confusion between the standard Church-Turing's Thesis which is widely accepted and its variants which are controversial. Analogous confusion is possible about the Martin-Löf-Chaitin's Thesis which is only for us the statement of the identification of a mathematical notion with an intuitive and metamathematical notion. We are not concerned in our discussion with the physical notion of chance or indeterminism, and we are not concerned with the problem of the possibility of free choice by a mind or a brain.

The Church-Turing's Thesis and the Martin-Löf-Chaitin's Thesis are not definitions. Really they are falsifiable and the proof of this is that the Thesis of Popper (which identifies random sequences with sequences without aftereffect) is now falsified by the results of [Ville 1939].

The Church-Turing's Thesis and the Martin-Löf-Chaitin's Thesis are really similar: each of them is a statement of identification of a mathematical notion with an intuitive metamathematical one: the first is about the notion of algorithm, the second is about the notion of chance. It is impossible to prove these theses because they are not mathematical results, but our previous informal notions are sufficiently precise so that the possibility of refutation still exists, and also that arguments may be given in favour or against these theses.

We have made a classification of the arguments about Church-Turing's Thesis and Martin-Löf-Chaitin's Thesis and a comparison as precise as possible.

(a) Arguments by means of examples.

(a1) Arguments for the Church-Turing's Thesis.

Usual functions as n -> 2n; n -> !; n -> n-th prime number and many others, are calculable in the intuitive metamathematical sense, and it is easy to prove that they are recursive, hence the Church-Turing's Thesis is not too restrictive.

(a2) Arguments for the Martin-Löf-Chaitin's Thesis.

Sequences as (0 0 0 0 ...) (0 1 0 1 0 1 ...) and many others are non-random in the intuitive metamathematical sense and it is easy to prove that they are not MLC-random, hence the Martin-Löf-Chaitin's Thesis is not too tolerant.

(a3) Comparison and remarks.

Arguments by means of examples tell us in the first case that the thesis is not too restrictive and in the second case that the thesis is not too tolerant. With a complementation (i.e. seeing the Martin-Löf-Chaitin's Thesis is

about non-random sequences) the two arguments are of equal strength. These arguments are arguments of **minimal adequation**: functions or sequences for which we have a clear obvious intuitive judgment are correctly classified by the mathematical notions.

Concerning the Church-Turing's Thesis the proposed examples are often proposed by infinite families (for example the family of polynomial functions) but always denumerable families, and no non-denumerable families of examples can be given (for the entire set of recursive functions is denumerable !) About the Martin-Löf-Chaitin's Thesis some non-denumerable sets of non-random sequences are easy to propose (for example the set of sequences verifying $x_{2n}=x_{2n+1}$ for every n). Hence it may be said that the Martin-Löf-Chaitin's Thesis is better supported by examples than the Church-Turing's Thesis.

It is clear that this type of arguments is not sufficient to reach a definitive judgment about the theses in question, but their importance is great and some examples have played an important role in the history of the definition of random sequences: the construction of sequences not satisfying the law of iterated logarithm and hence non-random (in the intuitive metamathematical sense) by Ville is what have proved that the definition of von Mises Church or of Popper had to be eliminated.

(b) Arguments by means of counterexamples.

(b1) Arguments for the Church-Turing's Thesis.

By diagonalization we can obtain functions which are not recursive and for which we have no reason to believe that they are intuitively calculable. This shows that the Church-Turing's Thesis is not too tolerant (and in particular that the thesis is not empty).

(b2) Arguments for the Martin-Löf-Chaitin's Thesis.

Using non-constructive arguments (about null set in measure theory), or using more direct argument (definition of the number omega of Chaitin [Chaitin 1987a] [Chaitin 1987b]) it can be proved that MLC-random sequences exist. The Martin-Löf-Chaitin's Thesis is not too restrictive.

(b3) Comparison and remarks.

As about the arguments by means of examples, within a complementation, it seems that the arguments by means of counterexamples are equivalent in strength with respect to Church-Turing's Thesis and Martin-Löf-Chaitin's Thesis. But the methods used to obtain MLC-random sequences are more sophisticated than those used to obtain non-recursive functions. Results about the incompressibility of random sequences show precisely that every MLC-random sequence is very hard to obtain: the first n bits of a random sequence cannot be computed by a program of length less than n. So random sequences much more than non-recursive functions are somewhat irreal, and so we can say that the Martin-Löf-Chaitin's Thesis is less supported by counterexamples than the Church-Turing's Thesis.

(c) Arguments based on the intrinsic convincing strength of the definitions.

(c1) Arguments for the Church-Turing's Thesis.

There are many definitions of the family of recursive functions. Each of them is based on a mathematical formalization of an informal notion of algorithmically calculable function. So each of them gives more or less the feeling that the idea of what is the intuitive notion of algorithmically calculable function is captured in the mathematical definition of recursive functions. Whatever mathematical formalisation you try for the notion of calculable function, it is now absolutely certain (in 1992) that you will obtain a definition easily provable equivalent to the others.

Hence the intrinsic convincing strength of all until today tried formalization of the notion of algorithmically calculable function is summed in favour of the Church-Turing's Thesis. There are several hundred such definitions, and their set is the more profound argument in favour of the Church-Turing's Thesis.

These definitions are based on moddisations of machines like Turing's machines or more complicated ones (Turing, Kolmogorof-Uspinski, Markov, Gandy), on the basis of consideration about deductions of values from systems of equations (Gödel, Kreisel-Tait), on formal systems for arithmetic (Gödel, Tarski), on grammar production rules (Post, Chomsky) and many others.

(c2) Arguments for the Martin-Löf-Chaitin's Thesis.

About the Martin-Löf-Chaitin's Thesis there are mainly 3 families of definitions. The first based on the notion of effective random tests (Martin-Löf). The second based on algorithmic information theory (Kolmogorof, Chaitin, Levin, Schnorr). The third based on imprevisibility and impossibility of winning gambler methods (Schnorr, Chaitin).

(c3) Comparison and remarks.

Each of the definitions of MLC-random sequences is interesting and gives a somewhat convincing element, but the chaotic history of the notion of random sequences is such that none of these definitons alone is sufficient to ensure us that we have handled the good notion. It's the opposite with the Church-Turing's Thesis for which one well-chosen definition can give a good definitive argument.

(d) Arguments of convergence of the definitions.

(d1) Arguments for the Church Turing's Thesis.

Each definition of recursive functions is in itself an interesting argument, but the fact that all the definitions are equivalent is surprising and indirectly constitutes the most convincing argument in favour of the Church-Turing's Thesis. Initially there may have been several notions of algorithms, even an infinite number of notions of algorithm and no good notions. The proof that all the attempts to make definitions gives us the same class of functions is a first order argument and it is an argument which likes the mathematician for it is a non trivial one (i.e. is based on a real mathematical work).

(d2) Arguments for the Martin-Löf-Chaitin's Thesis.

The same thing holds for Martin-Löf-Chaitin's Thesis, but with less strength. For there are less demonstrably equivalent definitions of the notion of random sequences, and also for the formulation of the good definition was preceded by a long period in which many bad thesis were proposed. But now the convergence argument is important and it is the very argument which was stressed in [Kolmogorof Uspenskii 1987]. Perhaps new definitions will be formulated which will prove equivalent to the Martin-Löf's definition.

(d3) Comparison and remarks.

Here we have a clear advantage in favour of the Church-Turing's Thesis over the Martin-Löf-Chaitin's Thesis for the first one is supported by several hundred equivalent definitions, meanwhile the second one has only several equivalent definitions and many non-equivalent definitions.

Please note that it is wrong to think that the convergence argument alone is a good argument. There are also many definitions of the class of primitive recursive functions (or of the class of the recursive sequences of 0 and 1), and these definitions cannot be considered as a good formalization of the notion of algorithmically calculable functions (or of the notion of non-random sequences).

(e) Arguments of robustness.

(e1) Arguments for the Church-Turing's Thesis.

The fact that it is impossible to diagonalize the class of recursive functions is what convinced Kleene in 1934 that the recently formulated proposition of Church was correct. This robustness of the class of recursive functions has many other aspects. For example all the variant notions of Turing machines (machines with several tapes, non deterministic machines, machines with a two-diminsional work-space, set of simultaneous communicating machines, etc.) give the good notion of recursive functions. This proves that the notion of recursive function is an intrinsic one, hence that it is an important one, hence that it is presumably the one expected.

(e2) Arguments for the Martin-Löf-Chaitin's Thesis.

The notion of random sequences resists also to small modifications in the formulatons of the definitions. One of the most remarkable is based on what Chaitin calls the "complexity gap": to define random sequences with the condition «$H(x_n) > n-c$», or with the condition «$H(x_n)-n$ tends to infinite» is equivalent. But there are many other examples of robustness of the definition of random sequences with regard to the random tests used int he Martin-Löf's definition.

(e3) Comparison and remarks.

Convergence and robustness show that the mathematical notions of recursive functions and random sequences are of the same nature that the number Π or the field of complex numbers: they are ubiquitous mathematical objects, and consequently are profound and important ones. Convergence and robustness are symptoms that we are right in identifying these notions with the intuitive metamathematical notions we are interested in.

Here there are anew more arguments of robustness in favour of the Church-Turing's Thesis than in favour of the Martin-Löf-Chaitin's Thesis.

About the questions of robustness it must be said that relatively to the finite there is no robustness of the notion of recursive function or of the notion of random sequence. Even if all the finite approximations f_n of a function f (functions f_n which are equal to f for every $m \leq n$) are recursive then the limit function f is not necessarily recursive. There is a similar result for random sequences.

(f) Arguments of duration, and of resistance to concurrent propositions.

(f1) Arguments for the Church-Turing's Thesis.

No truly concurrent theses have been formulated for identifying the notion of algorithmically calculable function. No truly good argument against the (standard) Church-Turing's Thesis had been proposed. These 50 years of success without real rival are a very strong argument in favour of the Church-Turing's Thesis. Perhaps every work on computer science may be considered as an indirect confirmation of it. Perhaps also the mathematical use of the Church-Turing's Thesis in order to shorten the proof in recursivility theory is an indirect argument, and certainly this shows the confidence of mathematicians in the Church-Turing's Thesis.

(f2) Arguments for the Martin-Löf-Chaitin's Thesis.

The Martin-Löf-Chaitin's Thesis is not attested by a similar long duration. Between 1966 and 1976, before the equivalences had been proved relating the definition in terms of effective statistical tests and the definition in terms of algorithmic information theory, the Martin-Löf-Chaitin's Thesis was uncertain. Some concurrent theses and in particular the proposition of Schnorr ([Schnorr 1971a] [Schnorr 1977]) are not absolutely eliminated today, they are only becoming less supported than the Martin-Löf-Chaitin's Thesis. The possibility of new mathematical results is always present, and a new evolution of the subject is not impossible. For example van Lambalgen says that he is «convinced that a satisfactory treatment of random sequences is possible only in set theory lacking the power set axiom, in which random sequences "are not already there"» and that non classical logic is the only way to obtain a definitive solution [van Lambalgen 1987].

Today the Martin-Löf-Chaitin's Thesis is supported by near all the specialists of the subject: Chaitin [Chaitin 1987b], Kolmogorof and Uspenskii [Kolmogorof Uspenskii 1987], Gacs [Gacs 1986], Schnorr (with a slight restriction [Schnorr 1977]), Levin [Levin 1984].

(f3) Comparison and remarks.

The Martin-Löf-Chaitin's Thesis is clearly less supported by this type of arguments than the Church-Turing's Thesis: it seems totally impossible that a new thesis replaces the Church-Turing one, it seems unlikely that a new thesis replaces Martin-Löf-Chaitin's one, but in each case the constructive mathematics have certainly something to say.

(g) Arguments of effectivity, fruitfulness and usefulness.

(g1) Arguments for the Church-Turing's Thesis.

Nothing can be proved true by saying that it is effective, fruitful or useful, but if a thesis is uninteresting, not applicable and does not give rise to new ideas we can imagine that the thesis is false and that we shall never see its falseness. Hence arguments of effectivity, fruitfulness and usefulness has to be considered.

The Church-Turing's Thesis is effective (it enables simplification of mathematical proof and hence allows to go further in the development of the theory of recursivity), fruitful (it gives a profound insight in our conception of the world), useful (in computer science for example the strength of programming language is studied, and a first step is always to prove that a language is algorithmically complete i.e. by using Church-Turing's Thesis that all recursive functions are programmable).

(g2) Arguments for the Martin-Löf-Chaitin's Thesis.

The Martin-Löf-Chaitin's Thesis is fruitful (see: [Chaitin Schwartz 1978] [Li Vitanyi 1989]) but it is so ineffective (in part due to strong incompleteness Gödel's Theorem that are related) that no concrete utilization of MLC-random sequences is possible. This problem is what has motivated new researches in the theory of pseudo-random sequences [Goldreich 1988].

The usefulness of the Martin-Löf-Chaitin's Thesis is also attested by recent uses of the algorithmic information theory in physics [Bennett 1988] in biology [Chaitin 1979] in statistics [Rissanen 1986] and in philosophy [Levin 1976a] [Levin 1984].

(g3) Comparison and remarks.

About this type of arguments the comparison is anew in favour of Church-Turing's Thesis, but we may see in a near future new developments and new utilizations of the concept of MLC-Random sequences.

5. CONCLUSION

The two thesis are really similar, the Church-Turing's Thesis offers a mathematical identification of the intuitive informal concept of algorithm, when the Martin-Löf-Chaitin's Thesis proposes an identification of the intuitive informal concept of randomness. They are profound insights in the mathematical and philosophical understanding of our universe. The first is more deeply attested and it has the advantage that it was formulated 50 years ago. The second one is more complicated (or seems so) and perhaps in 25 years when it is 50 years old, it will reached to a certainty similar to the other.

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