

A Counterexample Concerning Iteratively Generated Sequences

J. P. R. R. DELAHAYE*

20, rue Lucien Dupuis, 28500 Vernouillet, France

Submitted by R. Bellman

We present a counterexample concerning the cluster set of a sequence (x_n) iteratively generated by a relation of sort $x_{n+1} = f(x_n)$. This counterexample answers a question posed in 1970 by F. T. Metcalf and T. D. Rogers (*J. Math. Anal. Appl.* 31 (1970), 206-212).

1. INTRODUCTION

An article by Metcalf and Rogers [1] concluded with the following problem:

For any given continuous application T of the metric space Y into itself, and for any given point $x_0 \in Y$ is it true that:

$$\mathcal{L}' = T\mathcal{L}'?$$

\mathcal{L} stands for the set of cluster points of the sequence (x_n) defined by the recursive relation $x_{n+1} = f(x_n)$ from the point x_0 .

\mathcal{L}' stands for the derived set of \mathcal{L} (i.e., the set of accumulation points of \mathcal{L}).¹

The answer is negative and we propose in this paper a counterexample.

This counterexample could not be simple, because if $\mathcal{L}' \neq T\mathcal{L}'$ then \mathcal{L}' is infinite [1, Theorem 3]. The inclusion $\mathcal{L}' \subset T\mathcal{L}'$ is always satisfied [1, Lemme, p. 210] so the most simple schema we could hope for \mathcal{L}' was the following:

$$\begin{aligned} &\mathcal{L}' \text{ denumerable,} \\ &\mathcal{L}' = \{l_1, l_2, \dots, l_n, \dots\} \cup \{l\}, \quad \mathcal{L}'' = \{l\}, \\ &T\mathcal{L}' = \{l_0, l_1, l_2, \dots, l_n, \dots\} \cup \{l\}, \\ &Tl_{n+1} = l_n, \quad n \in \mathbb{N}^*. \end{aligned}$$

* Université des Sciences et Techniques de Lille, Lille, France.

¹ A detailed study of the properties of \mathcal{L} and various other counterexamples concerning those problems are given in [2].

This was possible and the space Y we are proposing is a subspace of \mathbb{R}^2 , hence verifying

$$Y' \neq \emptyset, \quad Y'' \neq \emptyset, \quad Y''' \neq \emptyset, \quad Y^{(4)} = \emptyset.$$

2. DESCRIPTION OF Y

Let

$$A = \left\{ \left(\frac{1}{2^n}, 0 \right) \mid n \in \mathbb{N} \right\} \cup \{ (0, 0), (3/2, 0) \},$$

$$B = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^m} \right) \mid m \in \mathbb{N}, n \in \mathbb{N}, m \geq n \right\},$$

$$C = \left\{ \left(\frac{3}{2}, \frac{1}{2^n} \right) \mid n \in \mathbb{N} \right\},$$

$$D = \left\{ \left(\frac{3}{2}, \frac{1}{2^n} + \frac{1}{2^m} \right) \mid n \in \mathbb{N}^*, m \in \mathbb{N}^*, n < m \leq 2n \right\},$$

$$E = \left\{ \left(\frac{1}{2^n} + \frac{1}{2^m}, \frac{1}{2^n} + \frac{1}{2^m} \right) \mid n \in \mathbb{N}^*, m \in \mathbb{N}^*, n < m \leq 2n \right\}.$$

For any point $X = (x, y)$ in the set $B \cup E$, we take a sequence $(X_n)_{n \in \mathbb{N}}$ of points in \mathbb{R}^2 , verifying:

- (i) all the X_n introduced in this operation are distinct,
- (ii) for any given $X = (x, y) \in B \cup E$ the set $\{X_n \mid n \in \mathbb{N}\}$ has a diameter $\leq y/4$, and (X_n) converges to X .

(This is to have the relation

$$\bigcup_{X \in B \cup E} \{X_n \mid n \in \mathbb{N}\}' = B \cup E \cup A - \{(\frac{3}{2}, 0)\}.)$$

It is possible to define the X_n , for example, in the following way:

if

$$X = \left(\frac{1}{2^p}, \frac{1}{2^q} \right) \in B,$$

$$X_n = \left(\frac{1}{2^p}, \frac{1}{2^q} \right)_n = \left(\frac{1}{2^p} + \frac{1}{2^{q+2+n}}, \frac{1}{2^q} \right);$$

if

$$X = \left(\frac{1}{2^p} + \frac{1}{2^q}, \frac{1}{2^p} + \frac{1}{2^q} \right) \in E,$$

$$X_n = \left(\frac{1}{2^p} + \frac{1}{2^q}, \frac{1}{2^p} + \frac{1}{2^q} \right)_n = \left(\frac{1}{2^p} + \frac{1}{2^q} + \frac{1}{2^{q+2+n}}, \frac{1}{2^p} + \frac{1}{2^q} \right).$$

Then we propose:

$$Y = A \cup B \cup C \cup D \cup E \cup \bigcup_{X \in B \cup E} \{X_n \mid n \in \mathbb{N}\}.$$

F is closed in \mathbb{R}^2 , because any convergent sequence of points of Y possesses its limit in F . With the distance of \mathbb{R}^2 (e.g., $d((x, y), (x', y')) = ((x - x')^2 + (y - y')^2)^{1/2}$), F is a compact metric space.

3. DESCRIPTION OF T

The application T is defined in the following manner:

On A

$$\begin{aligned} T(0, 0) &= (0, 0), \\ T(3/2, 0) &= (0, 0), \\ T(1/2^{m+1}, 0) &= (1/2^m, 0), \quad m \in \mathbb{N}, \\ T(1, 0) &= (3/2, 0). \end{aligned}$$

On B

$$\begin{aligned} T\left(1, \frac{1}{2^m}\right) &= \left(\frac{3}{2}, 0\right), \quad m \in \mathbb{N}, \\ T\left(\frac{1}{2^n}, \frac{1}{2^m}\right) &= \left(\frac{1}{2^{n-1}}, \frac{1}{2^m}\right), \quad n \in \mathbb{N}^*, \quad m \in \mathbb{N}^*, \quad m \geq n, \end{aligned}$$

On C

$$T\left(\frac{3}{2}, \frac{1}{2^n}\right) = \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}, \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}\right)_0, \quad n \in \mathbb{N}.$$

On D

$$\begin{aligned} T\left(\frac{3}{2}, \frac{1}{2^n} + \frac{1}{2^m}\right) &= \left(\frac{1}{2^n} + \frac{1}{2^{m+1}}, \frac{1}{2^n} + \frac{1}{2^{m+1}}\right)_0, \\ & \quad n \in \mathbb{N}^*, \quad m \in \mathbb{N}^*, \quad n < m \leq 2n - 1, \\ T\left(\frac{3}{2}, \frac{1}{2^n} + \frac{1}{2^{2n}}\right) &= \left(\frac{1}{2^n}, \frac{1}{2^n}\right)_0, \quad n \in \mathbb{N}^*. \end{aligned}$$

On E

$$T\left(\frac{1}{2^n} + \frac{1}{2^m}, \frac{1}{2^n} + \frac{1}{2^m}\right) = \left(\frac{1}{2^{n-1}} + \frac{1}{2^{m-1}}, \frac{1}{2^{n-1}} + \frac{1}{2^{m-1}}\right),$$

$$n \in \mathbb{N}^*, \quad m \in \mathbb{N}^*, \quad 1 < n < m \leq 2n - 1,$$

$$T\left(\frac{1}{2^n} + \frac{1}{2^{2n}}, \frac{1}{2^n} + \frac{1}{2^{2n}}\right) = \left(\frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}\right), \quad n \in \mathbb{N}^*.$$

On $\bigcup_{X \in B} \{X_n \mid n \in \mathbb{N}\}$

$$T\left(1, \frac{1}{2^m}\right)_0 = \left(\frac{3}{2}, \frac{1}{2^m}\right), \quad m \in \mathbb{N},$$

$$T\left(1, \frac{1}{2^m}\right)_p = \left(\frac{3}{2}, \frac{1}{2^{m+p}} + \frac{1}{2^{2m+p+1}}\right), \quad m \in \mathbb{N}, \quad p \in \mathbb{N}^*,$$

$$T\left(\frac{1}{2^n}, \frac{1}{2^m}\right)_p = \left(\frac{1}{2^{n-1}}, \frac{1}{2^m}\right)_p, \quad n \in \mathbb{N}^*, \quad m \in \mathbb{N}, \quad m \geq n, \quad p \in \mathbb{N}.$$

On $\bigcup_{X \in E} \{X_n \mid n \in \mathbb{N}\}$

$$T\left(\frac{1}{2^n} + \frac{1}{2^{2n}}, \frac{1}{2^n} + \frac{1}{2^{2n}}\right)_p = \left(\frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}\right)_{p+1}, \quad n \in \mathbb{N}^*, \quad p \in \mathbb{N},$$

$$T\left(\frac{1}{2^n} + \frac{1}{2^m}, \frac{1}{2^n} + \frac{1}{2^m}\right)_p = \left(\frac{1}{2^{n-1}} + \frac{1}{2^{m-1}}, \frac{1}{2^{n-1}} + \frac{1}{2^{m-1}}\right)_{p+1},$$

$$n \in \mathbb{N}^*, \quad m \in \mathbb{N}^*, \quad 1 < n < m \leq 2n - 1, \quad p \in \mathbb{N}.$$

To verify that T is continuous, we consider all converging sequences $(Z_n)_{n \in \mathbb{N}}$ in Y and we discover that

$$\lim_{m \rightarrow \infty} TZ_m = T \lim_{m \rightarrow \infty} Z_m.$$

Y is a metric space so this is sufficient to obtain continuity of T .

4. SEQUENCE $(x_n)_{n \in \mathbb{N}}$

The sequence $(x_n)_{n \in \mathbb{N}}$ is then defined from the point $x_0 = (1, 1)_0$. (See Fig. 1.)

T is constructed to have: $\{x_n \mid n \in \mathbb{N}\} = \bigcup_{X \in B \cup E} \{X_n \mid n \in \mathbb{N}\} \cup D \cup C$.
So we have:

$$\mathcal{L} = B \cup E \cup A,$$

$$\mathcal{L}' = A - \left\{\left(\frac{3}{2}, 0\right)\right\},$$

$$T\mathcal{L}' = A.$$

And then: $\mathcal{L}' \neq T\mathcal{L}'$.

REFERENCES

1. F. T. METCALF AND T. D. ROGERS, The cluster set of sequences of successive approximations, *J. Math. Anal. Appl.* **31** (1970), 206–212.
2. J. P. R. R. DELAHAYE, “Quelques Problèmes posés par les Suites de Points non Convergentes et Algorithmes pour traiter de Telles Suites,” Thèse, Université de Lille, 1979.