A computational definition of financial randomness

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Under the Efficient Market Hypothesis (EMH), in a one security and cash trading system, no one can outperform the buy-and-hold strategy in the long run. This classical proposition calls for a formal definition of outperforming trading rules. This article proposes a computational approach that completely departs from the probabilistic framework in which the question was originally formulated. Inspired by Schnorr’s definition of binary random string (Schnorr [Math. Syst. Theory, 1971, 5, 246–258]), computable functions are used to model effective trading rules that can be applied to financial price series. In the absence of transaction costs, a price sequence is said to be ‘unbeatable’, if no effective trading rule can generate indefinitely more profits than its buy-and-hold alternative. As a quantitative formulation of recent thoughts on EMH (e.g. Lo [J. Portfolio Manage., 2004, 30, 15–20], Kandhuni and Lo [J. Invest. Manage., 2007, 5, 29–78], Hasan hodzic et al. [Quant. Financ., 2011, 11(7), 1043–1050]), our definition reconsiders the notion of financial randomness and reconciles EMH with the performance of ‘constantly re-balanced portfolios’ and ‘automated trading systems’ (Cover [Math. Financ., 1991, 1(1), 1–29], Cover and Ordentlich [IEEE Trans. Inf. Theory, 1996, 42, 348–363], Fagiuoli et al. [Quant. Financ., 2010, 10(4), 401–420]). Thus, we propose a ‘computational definition of financial randomness’ in formulating the concept of ‘unbeatable price sequences’.

Keywords: Computational finance; Computability theory; Market outperforming; Market efficiency

JEL Classification: C63, G00

1. Introduction

On 9 March 2009, the Dow Jones Industrial Average (DJ) closed at a new 12-year low of 6547 points, that is, less than half of its value in October 2007, when the index reached 14164 points. One year later, in April 2010, the DJ recovered more than 60% of this historic drop on turning around 11 000 points. During this three-year period, a good market timing strategy could have (at least) doubled its initial capital on buying the bottom and selling the top. However, very few investors were able to extract this potential benefit from that extraordinary variation so that an exploitable trading rule based on the forecasts appears questionable for the data-set (Lane et al. 1996, p. 424). Sarantis (2001) verified the capacity of STAR models to predict the sign of market indexes’ variations and reported a success rate (or ‘hit-rate’) smaller than 60%, which, according to Grabbe (1996), is not sufficient to construct a profitable market timing strategy. In a similar way, Dunis et al. (2003) tested the performance of ARMA, MA, Logit and Artificial Neural Networks models over currency data without being able to report hit scores greater than 57.24%. Altogether, these results suggest that neither linear nor non-linear models can succeed in predicting the market as to generate profitable trading rules (see Tsay 2005).

Nevertheless, until now, no real consensus has been established on this subject. We identify (at least) three steps in the evolution of the EMH literature highlighting this debate.

(1) Some so-called market anomalies seem to indicate the deficiency of the EMH. For example, ‘buying loser...
and selling winner’ strategies based on the ‘contrarian effect’ De Bondt and Thaler (1985) or ‘buying winner and selling loser’ strategies based on the ‘momentum effect’ Jegadeesh and Titman (1993, 2001) are often said to offer better performances than the market portfolio even after transactions costs Korajczyk and Sadka (2004), Boynton and Oppenheimer (2006). However, these anomalies are generally characterized by cross-sectional tests. As these tests use the Market Portfolio as the only benchmark to be outperformed, they rely on the CAPM, which is itself subject to criticisms (see, for example, Fama and French 1993). In a similar perspective, one can also refer to the works of Caginalp and Laurent (1998). Using a more general information base, some strategies seem to deliver interesting performances as well (Duran and Caginalp 2007, Duran and Bommarito 2011 and Shen 2003).

(2) More recently, the performance of ‘constant re-balanced portfolios’ Cover (1991), Cover and Ordentlich (1996), Fagiuoli et al. (2007) has also provoked a fair amount of debates on the EMH. The composition of these portfolios is continuously rebalanced through sophisticated programming techniques that exploit a wide range of patterns. For example, Creamer and Freund (2010) propose a trading system that is capable of generating abnormal return on the SP500 during the period 2003-2005. See also Shiryaev et al. (2008) in the same stream of literature.

(3) This empirical trend finds an echo in the literature on market regime switching in time. Fabozzi (2008) sums up this idea quoting Andrew Lo: According to Lo, markets are adaptive structures in a state of continuous change; profit opportunities disappear as agents learn, but they do not disappear immediately and can, for a while, be profitably exploited. In the meantime, new strategies are created and along with them new profit opportunities Fabozzi (2008, p. 650).

If patterns are modified by models that try to exploit them, constant adaptation is necessary for all quantitative trading firms. This ‘technological arms race Hasanhodzic et al. (2011, p. 1043)’ will result in a form of ‘technical market efficiency’ in the sense that everyone will have a hard time updating models just in time so to outperform the buy-and-hold in the long run. In other words, EMH should be reconsidered in a dynamic framework that allows short-term patterns.

Following these ideas, we define a notion of financial randomness embedded in a computational framework: financial dynamics will be said to be ‘efficient’ if no calculable trading function can beat the buy-and-hold alternative. In this, we follow in the footsteps of Hasanhodzic et al. (2011), who made a first attempt in this direction. However, Hasanhodzic et al. (2011) adopted a series of probabilistic formulations that hindered the use of calculability notions in their definition.

To a certain extent, our work also contributes to computational economics Velupillai (2000) in formulating a definition of financial randomness using concepts initially proposed in algorithmic complexity.

This paper falls therefore into a tradition initiated by Von Mises, who first challenged the scientific community to formulate a definition of binary random strings. In response to this challenge, three equivalent formulations were proposed by Martin-Löf (1966), Chaïtin (1969), Schnorr (1971) and Levin (1977) (see Nies (2009), Downey and Hirschfeldt (2010), especially chapter 7). One of the most remarkable conclusions on this topic is the impossibility of formulating such a definition using the extraction-based approach proposed by Von Mises and followed by Church (1936).

We follow the ideas of Schnorr (1971), Ville (1939) and Shafer and Vovk (2001), who exploit the relationship between a random binary string and the maximum profits one can obtain from it with a martingale. However, instead of focusing on binary strings, we generalize this relation to integer strings as they better represent successive financial prices. More precisely, we propose a formal definition of an unbeatable price sequence under the assumption of zero transaction costs. This definition allows us to investigate the possibilities for several effective strategies (i.e. those that can be automated with a computer) to outperform the buy-and-hold alternative in an accurate, but a very general sense.

To make a long story short, we develop a ‘computational definition of financial randomness’ in formulating the concept of ‘unbeatable price sequences’.

The paper is organized as follows. In section 2, we introduce the definition of unbeatable price sequences within a buying all and selling all framework that constrains trading strategies. In section 3, which deals with partial buying and selling operations, we show that geometric strategies can outperform the buy-and-hold alternative even if future returns are unpredictable. Finally, in section 4, we discuss and conclude our attempt to define financial randomness from a theoretical computer science perspective.

2. Outperforming the market within an ‘all-or-nothing’ framework

To define ‘outperforming the market’, one should first model trading rules and their corresponding performances. For the sake of simplicity, an ‘all-or-nothing’ framework is introduced in this section to represent investment strategies that are accessible to investors. Basically, a strategy seeks to identify signals indicating when to buy and when to sell. In the latter framework, investors follow their signals with quantities such that they either have no cash left after buying or no stock holdings left after selling. Partial buying and selling will be covered in the next section within a generalized framework.

2.1. The initial model

Given a sequence of financial commodity prices, a trading rule can be modelled as follows:‡

‡Lo (2004) referred to the Adaptive Market Hypothesis instead of EMH.

Brandouy et al. (2009) give concrete examples of buying all and selling all trading rules described in this section.
At stage $n$, sufficient liquidity at these prices is assumed. Framework, i.e., investors’ decisions cannot affect these prices and can be a stock, a portfolio or a market index. We posit a price-taker for simplicity of presentation, let $w_n$ denote the set of all rational numbers. $Q^+$ be the amount of available cash at time $n$. By definition, $S_0 \in \mathbb{N}$, which means that the starting capital cannot be negative or null. Let $a_n$ be the number of risky assets at time $n$. Without loss of generality, we impose $a_0 = 0$. Let $d_n(p_0, p_1, p_2, \ldots, p_{n-1})$ be a trading rule satisfying:

$$d_n(p_0, p_1, p_2, \ldots, p_{n-1}) \in \{0, -1, +1\}$$

and $a_{n-1} > 0$ is a necessary condition for $d_n(p_0, p_1, p_2, \ldots, p_{n-1}) = -1$.

where

- ‘$-1$’ means ‘convert all available assets into cash’,
- ‘$+1$’ means ‘convert all available cash into assets’,
- ‘0’ means ‘do nothing’.
- the necessary condition for “$d_n(p_0, p_1, p_2, \ldots, p_{n-1}) = -1$”, makes it impossible to short sell.

For simplicity of presentation, let $d_n$ denote $d_n(p_0, p_1, \ldots, p_{n-1})$. The relationship between $d_n, s_n$ and $a_n$ can be described as follows:

- when $n = 1$, $d_1$ takes a decision according to $p_0$. As all trading rules begin with cash, we get

$$s_1 = s_0 - d_1 s_0, \quad a_1 = a_0 + d_1 \frac{s_0}{p_0} \quad d_1 = d_1(p_0) \in \{0, +1\}$$

(1)

- when $n = 2$, we have

$$\begin{cases} s_2 = a_1 p_1, \quad a_2 = 0 & \text{if } d_2 = -1 \\ s_2 = s_1 - d_2 s_1, \quad a_2 = a_1 + d_2 \frac{s_1}{p_1} & \text{if } d_2 = 0, +1 \end{cases}$$

(2)

- At stage $n$, $d_n, s_n$ and $a_n$ verify:

$$\begin{cases} s_n = a_{n-1} p_{n-1}, \quad a_n = 0 & \text{if } d_n = -1 \\ s_n = s_{n-1} - d_n s_{n-1}, \quad a_n = a_{n-1} + d_n \frac{s_{n-1}}{p_{n-1}} & \text{if } d_n = 0, +1 \end{cases}$$

(3)

At stage $n$, wealth generated by $d_n$, denoted by $w_n$, can be calculated by

$$w_n = s_n + a_n \times p_n$$

(4)

Combining (3) and (4), $w_n$ is a function of $d_n, s_0$ and $p_0$.

Meanwhile, the buy-and-hold strategy generates a wealth of:

$$w_n^* = \frac{p_n}{p_0} s_0$$

(5)

Comparing $w_n$ with $w_n^*$, we can formulate a definition for outperforming the market as follows:

**Outperforming the market:** 2.1. A trading rule $d_n$ is said to **outperform** the price sequence $p_0, p_1, p_2, \ldots, p_n, \ldots$ i.f.f.:

$$\lim_{n \to \infty} \frac{w_n}{w_n^*} = \infty$$

(6)

$\dagger$There is no restriction on the nature of the risky asset $A$, which can be a stock, a portfolio or a market index. We posit a price-taker framework, i.e., investors’ decisions cannot affect these prices and sufficient liquidity at these prices is assumed.

$\ddagger\mathbb{Q}$ denotes the set of all rational numbers.

In our model, the trading function’s field is fixed to be integer or rational numbers. This choice makes it easy to use theoretical concepts of computability theory. Another solution could be to use computable real numbers, as proposed by Oliver Albert and applied to data analysis on physics by Pour-El and Richards (1989), for example. However, we chose not to do this to avoid unnecessary complications. What is more, from a practical point of view, and contrary to physics, financial price can never be truly continuous, particularly because of the presence of tick size.

Non-standard analysis approaches, e.g. Väth (2007), can also be considered to define the trading function $d_n$, however, this would lead our work towards an extremely inaccessible theoretical framework without adding conclusive value to the current approach.

2.2. **Computational framework vs. probabilistic tradition**

In this section, we try to establish a link between definition 2.1 and several probabilistic traditions in finance. First, we clarify how definition 2.1 takes into account the notion of financial risks; second, we show the difference between the limit condition (cf. equation (6)) and the classical criterion in terms of expected return.

2.2.1. **Computational definition and financial risks.** The difference between an active strategy $d_n$ and the buy-and-hold alternative is that $d_n$ leaves the market from time to time. During those periods when $d_n$ keeps cash, its return is simply equal to 0, and so is its risk. Thus, without leverage effect, whatever $d_n$ decides to do, its risk is always smaller than that of the buy-and-hold alternative. In other terms, in a single security and cash system, the buy-and-hold strategy is exposed to the highest risk level. Therefore, to compare an active strategy to the latter benchmark, we can focus on accumulated profits only.

2.2.2. **Computational definition and expected returns.** Traditionally, financial researchers consider a strategy, $d_n$, to be better than the buy-and-hold alternative (denoted by $d_n^*$), if $d_n$’s expected return - on supposing its existence - is greater than that of $d_n^*$. From an empirical point of view, this condition can be translated into limit means: $d_n$ is better than $d_n^*$, if its limit geometric mean is larger than that of $d_n^*$:

$$\lim_{n \to \infty} \sqrt[n]{w_n} > \lim_{n \to \infty} \sqrt[n]{w_n^*}$$

(7)

Since (7) implies (6), the latter is not stricter than the usual concept of ‘outperforming the market’ in Finance.

However, note that this implication does not hold in the opposite sense. The following developments prove this assertion.

Let $p_0, p_1, p_2, \ldots, p_n, \ldots$ be a price sequence satisfying

$$\begin{cases} p_{n+1} = \frac{1}{2} p_n \quad n = m^2, m \in \mathbb{N} \\ p_{n+1} = 2 p_n \quad n \neq m^2 \end{cases}$$

(8)

In other words, we consider here a fast growing series of prices with drops that become more infrequent with time.
Figure 1. Difference between definition 2.2 and the traditional geometric mean criterion. (a) Case 1: as shown in proposition 2.2, although \( d_n \) (represented by the black curve) does a better job than buy-and-hold (represented by the grey one), it cannot be said to outperform the latter if one adopts the geometric mean criterion specified in equation (7). However, according to definition 2.2, \( d_n \) actually outperforms the buy-and-hold alternative since it generates infinitely more profits than the benchmark. See also Figure 2. (b) Case 2: \( d_n \)'s performance on \( p_0, p_1, p_2, \ldots, p_n, \ldots \) is represented by the black line and that of buy-and-hold by the grey one. \( d_n \) outperforms buy-and-hold under the geometric mean condition and in the limit gain condition.

Figure 2. Difference between definition 2.2 and the traditional geometric mean.

**Proposition 2.2.** The strategy \( d_n \), holding cash when \( n = m^2 \) and assets at other moments, will outperform the price sequence \( p_0, p_1, p_2, \ldots, p_n, \ldots \). Moreover, profits generated by \( d_n \) satisfy

\[
\lim_{n \to \infty} \sqrt[n]{\frac{w_n}{w_0}} = \lim_{n \to \infty} \sqrt[n]{\frac{w_n^*}{w_0}} = 2
\]  

(9)

**Proof.** As the difference between \( d_n \) and buy-and-hold depends on the moments where \( d_n \) holds cash, we get:

\[
\log \left( \frac{w_n}{w_n^*} \right) = -\log \left( \sum_{i=1}^{n} \frac{p_i}{p_{i-1}} \right) = m = \lfloor \sqrt{n} \rfloor
\]

(10)

Here, \( \lfloor \sqrt{n} \rfloor \) means the biggest integer not exceeding \( \sqrt{n} \). ‘log’ is here the binary logarithm function.\(^\dagger\)

Since \( \frac{w_n}{w_n^*} \to \infty \) when \( n \to \infty \), \( d_n \) outperforms the price sequence \( p_0, p_1, p_2, \ldots, p_n, \ldots \).

At the same time,

\[
\lim_{n \to \infty} \sqrt[n]{\frac{w_n}{w_n^*}} = \lim_{n \to \infty} \frac{\lfloor \sqrt{n} \rfloor}{w_0} = 1
\]

(11)

In other words, equation (11) says that the geometric mean of the difference in gains between \( d_n \) and \( d_n^* \) is equal to one. With \( w_0, w_n^* > 0 \), equation (11) can be transformed into:

\[
\lim_{n \to \infty} \sqrt[n]{\frac{w_n}{w_0}} = \lim_{n \to \infty} \sqrt[n]{\frac{w_n^*}{w_0}} = 2
\]

(12)

\(\square\)

According to this proposition, a strategy \( d_n \) can outperform a price sequence without generating a bigger limit mean than the buy-and-hold alternative. While the difference between \( d_n \) and \( d_n^* \) (or \( w_n/w_n^* \)) regularly increases with \( n \), its growth rate is too slow\(^\dagger\) to be observed in the limit geometric mean. As illustrated in figure 1, according to our definition, \( d_n \) is said to ‘outperform’ the buy-and-hold in both illustrated cases (see figure 1(a) and 2(b)).\(§\) However, under the geometric mean criterion, \( d_n \) from case 1 cannot outsmart the buy-and-hold, since its profits do not increase quickly enough. Their growth rate is illustrated in Figure 2, where we can clearly see that the marginal growth of \( \log(w_n/w_n^*) \) tends towards 0 while \( n \) increases for case 1 and remains constant for case 2.

So, definition 2.1 is not equivalent to the financial convention represented by equation (7) in the sense that the latter equation is stricter than the proposed definition. Moreover, our definition has a more general sense as it does not suppose the existence of \( \lim_{n \to \infty} \sqrt[n]{\frac{w_n}{w_0}} \) and \( \lim_{n \to \infty} \sqrt[n]{\frac{w_n^*}{w_0}} \).

From a mathematical point of view, the probabilistic tradition in finance actually imposes two conditions on the concept of outperforming the market:

- \( d_n \) generates indefinitely more profits than \( d_n^* \).
- \( w_n/w_n^* \) grows in an exponential way, more precisely, \( w_n/w_n^* > \lambda^n \), with \( \lambda > 1 \).

However, according to our definition, a strategy \( d_n \) outperforms the buy-and-hold alternative only if it generates indefinitely more profits than \( d_n^* \). Logically, the second condition cited above is an extra condition without theoretical support, as it

\(\dagger\)Throughout this article, except if specifically indicated, log will always denote the binary logarithm.

\(\dagger\)In our example, the \( \log(w_n/w_n^*) \) is added by \( '1' \) at each time \( \lfloor \sqrt{n} \rfloor \) increases. As \( \lim_{n \to \infty} \sqrt[n]{\frac{w_n}{w_0}} = 0 \), this difference - although tending to \( \infty \) with \( n \) - cannot modify the limit mean of \( w_n \).

\(§\)Differences between these cases are described in the figure comments.
is really difficult to explain why $d_n$ in figure 1(a) cannot be considered to have outperformed the buy-and-hold alternative. In these figures, the Y-axis represents profits in logarithmic scale, and the X-axis the number of simulated periods. In figure 1(a), we simulated a strategy $d_n$ on price sequence $p_0, p_1, p_2, \ldots, p_n, \ldots$ with $d_n$ and $p_0, p_1, p_2, \ldots, p_n, \ldots$ defined such as in proposition 2.2. In figure 1(b), $p_0, p_1, p_2, \ldots, p_n, \ldots$ is defined by equation (13), and $d_n$ is a strategy holding cash when $n = 100 m$, and assets otherwise.

\[
\begin{cases}
    p_{0} = \frac{1}{2} n = 100 m, m \in \mathbb{N}^* \\
    p_{0} = 2 n \neq 100 m
\end{cases}
\] (13)

The notion of ‘effective trading rule’ must be carefully distinguished from the idea of ‘tractable investment strategy’. To a certain degree, the term ‘tractable’ is another way to address the question of computability and effectivity (i.e. including constraints on the computing time). All tractable strategies are computable functions, but all computable functions are not tractable, as computability only requests the possibility to reach a solution by a Turing machine, and this without any constraint on the computing time. In this paper, we explicitly make the choice of computability instead of tractability to condition the decision function $d_n$ for two reasons:

- To the best of our knowledge, the maximum executing time of a tractable strategy has never been precisely formulated.
- Executing time to reach a solution for $d_n$ not only depends on the function itself, but also on the design of the considered Turing machine.

This paper is thus about the possibility to beat a price series that, logically, should not depend on the mobilized Turing machine. Thus, we do not restrict our analysis to a specific tractable strategy such as a constantly rebalanced portfolio (even if we use it for demonstration purposes in proposition).

Then, we define:

**Unbeatable sequences** 2.4. A price sequence $p_0, p_1, p_2, \ldots, p_n, \ldots$ is said to be ‘unbeatable’ if no effective trading rule can outperform it.

Unbeatable sequences in this particular sense exist. For example, the (improbable) price sequence 10, 12, 10, 12, 10, 10, 10, … is unbeatable by the effective rule ‘buy (sell) at odd (even) steps’.

Unbeatable strings also exist. To beat buy-and-hold, we have to predict future drops. Thus, increasing sequences (with no drop to avoid) are always unbeatable. Consequently, buy-and-hold is the best strategy at each step. For example, the sequence 0, 1, 2, 3, 4, 5, 6, … is unbeatable, because there is no drop to be avoided.

Given the relation between a price sequence and a return series, definition 2.4 can be expressed in terms of return series as follows:

Let $r_0 = p_0$ and $r_n = \log(p_n) - \log(p_{n-1})$, we get a strict correspondence between the return series $r_0, r_1, r_2, \ldots, r_n, \ldots$ and the price sequence $p_0, p_1, p_2, \ldots, p_n, \ldots$. We then define:

**Unbeatable return series** 2.5. A return series is ‘beaten’ by an effective trading rule $d_n$, if its corresponding price sequence is beaten by $d_n$.

A return series is unbeatable, if its corresponding price sequence is unbeatable.

Based on this strict correspondence between unbeatable return series and price strings, running a trading rule over a given price sequence can be expressed in terms of successive returns.

Running a strategy $d_n$ on a given price sequence consists of separating its return series into two sets: the set of refused returns and that of used returns.

‘Used Returns’ and ‘Refused Returns’ 2.6. A return $r_n$ is said to be ‘used’ (‘refused’) by $d_n$ if $a_n > 0$ (resp. $a_n = 0$).
As the difference between $d_n$ and buy-and-hold depends on the sum of refused returns in the sense that

$$\log \left( \frac{w_n}{w_n^*} \right) = \sum_{i=0}^{n} -r_i$$

(14)

in each beatable sequence, there should be enough negative returns to make the sum $\sum_{i=0}^{n} -r_i$ tend to $\infty$. That is to say,

**Proposition 2.7** If $p_0, p_1, p_2, \ldots, p_n, \ldots$ is a beatable price sequence, then its corresponding returns, $r_0, r_1, r_2, \ldots, r_n, \ldots$, satisfy

$$\sum_{i=0, r_i<0}^{\infty} r_i = -\infty$$

(15)

For example, the return series consisting of an infinite succession of $1$, $\{1, 1, \ldots\}$ is unbeatable, since no subset in this series has a negative sum. On the contrary, the return series alternating $1$ and $-1$, $\{1, -1, 1, -1, \ldots\}$ is beatable since the effective strategy 'buy at even and sell at odd steps' uses all negative returns whose sum tends to $-\infty$.

Till now, examples of unbeatable return series are only positive ones (see the beginning of section 2.1). If unbeatable series were always positive, our definition would be of no interest. Hence, we show that there are unbeatable series satisfying equation (15). In the following developments, return series will be constructed based on Martin-Löf random strings (see Martin-Löf (1966)). In the latter, we replace $0$ by $-1$ and $1$ by $2$:

**Proposition 2.8** Let $a_0, a_1, \ldots, a_n, \ldots$ be a Martin-Löf random string.

Let $r_0, r_1, \ldots, r_n, \ldots$ be the return series constructed from $a_0, a_1, \ldots, a_n, \ldots$ on substituting '$0$' by $-1$ and '$1$' by $2$.

Then, $r_0, r_1, \ldots, r_n, \ldots$ is an unbeatable sequence fulfilling equation (15).

**Proof** Since there is an infinite number of $0$ in a Martin-Löf random string, the return sequence $r_0, r_1, \ldots, r_n, \ldots$ fulfills equation (15).

To show that the latter sequence is unbeatable, we propose a proof by contradiction. Suppose that $d_n$ is an effective trading rule outperforming the return sequence $r_0, r_1, \ldots, r_n, \ldots$. Let $a_n$ be the function that gives the number of assets held by $d_n$ at each step, as defined page 2.

Then, refused returns of $d_n$ should satisfy:

$$\lim_{n \to \infty} \sum_{i=0, a_i=0}^{n} -r_i = \infty$$

(16)

which implies

$$\lim_{n \to \infty} \sum_{i=0, a_i=0}^{n} r_i = -\infty.$$

(17)

As every effectively extracted subsequence of $r_0, r_1, \ldots, r_n, \ldots$ should contain 50% of $2$ and 50% of $-1$ at infinity, equation (17) cannot hold. □

### 3. A general trading model

In this section, the buying all and selling all strategies defined in section 2 will be generalized on permitting partial buying or selling at each stage. This general model is a better description of financial markets, as investors do vary their position in risky assets from time to time.

Let simple strategies denote buying all and selling all trading rules, and general strategies denote partial buying and partial selling rules. We will show that general strategies can beat more sequences than simple ones. For example, geometric strategies can beat the buy-and-hold even if the return series is totally unpredictable.

- Let $p_0, p_1, \ldots, p_n, \ldots$ be a general strategy delivering the rate of risky asset to be held in portfolio at the end of step $n$.
- Let $q_n (p_0, p_1, \ldots, p_{n-1}) \in \mathbb{Q}$ (denoted by $q_n$) be the same rate as previously but at the beginning of step $n$.

$d_n, q_n$ verify the following relation:

- Since $d_n$ begins with cash, we get $l_0 > 0, d_0 = q_0 = q_1 = 0$.
- At stage 1, $d_1$ makes a decision according to $p_0$.

$$q_1 = 0, \quad d_1 = q_0, \quad 0 \leq d_1 \leq 1$$

(18)

- At stage 2,

$$q_2 = \frac{d_1 p_1}{d_1 p_1 + (1 - d_1)} d_2 \in \mathbb{Q}, \quad 0 \leq d_2 \leq 1$$

(19)

- At stage $n$,

$$q_n = \frac{d_{n-1} p_{n-1}}{d_{n-1} p_{n-1} + (1 - d_{n-1})} d_n \in \mathbb{Q}, \quad 0 \leq d_n \leq 1$$

(20)

So $q_n$ is computable if and only if $d_n$ is computable. Wealth generated by $d_n$ can then be calculated by:

$$w_n = l_0 \prod_{i=1}^{n} (d_i 2^{r_i} + 1 - d_i)$$

That of buy-and-hold is given by:

$$w_n^* = l_0 2^{\sum_{i=1}^{n} r_i}$$

(22)

This general model offers the possibility to vary one’s position in risky assets at each stage. Thanks to this possibility, ‘geometric strategies’, initially proposed by Bernoulli (1738) in a totally different context, can be compared to ‘buy-and-hold’. We will show in the following propositions that general trading rules can beat more sequences than simple ones. In the following developments, we borrow from section 2.8 (page 10) the same kind of construction for our return series.

**Proposition 3.1** Some unbeatable price sequences by simple trading rules can be outperformed by general ones.
Proof  Let \( a_0, a_1, \ldots, a_n, \ldots \) be a Martin-Löf random string where ‘0’ are replaced by ‘−1’. Without loss of generality, we impose \( a_0 = 1 \). Let \( r_0, r_1, \ldots, r_n, \ldots \) be a return series obtained from \( a_0, a_1, \ldots, a_n, \ldots \) on substituting all \( a_{2m+1}, m \in \mathbb{N}^+ \) by ‘1’. It will be shown that \( r_0, r_1, \ldots, r_n, \ldots \) is unbeatable by simple strategies, but beatable by geometric trading rules.

Let \( p_0, p_1, \ldots, p_n, \ldots \) be the price sequence corresponding to \( r_0, r_1, \ldots, r_n, \ldots \). Since \( r_n = 1 \) for all \( n = 2m + 1 \), \( p_0, p_1, \ldots, p_n, \ldots \) is a price sequence satisfying \( p_n \in \mathbb{N}^* \).

We show this proposition in two steps:

1. \( p_0, p_1, \ldots, p_n, \ldots \) is unbeatable by simple strategies.
   Suppose that there is an effective trading rule \( d_n \) which beats \( p_0, p_1, \ldots, p_n, \ldots \). Returns refused by \( d_n \), denoted \( r_0', r_1', \ldots, r_n', \ldots \), should satisfy:

   \[
   \lim_{n \to \infty} \sum_{i=0}^{n} -r_i' = \infty \tag{23}
   \]

   This implies that the limit frequency of ‘−1’ should be bigger than ‘1’ in the subsequence \( r_0', r_1', \ldots, r_n', \ldots \). As \( r_0, r_1, \ldots, r_n, \ldots \) is Martin-Löf random, none of its effectively extracted subsequence can satisfy equation (23).

2. \( r_1, r_2, \ldots, r_n, \ldots \) is beatable by a general strategy.
   Let \( d_n \) be a geometric strategy† that invests \( f \) percent of its wealth at even stages and 100% at odd ones. The growth rate of the wealth difference between \( d_n \) and buy-and-hold is given by:

   \[
   \lim_{n \to \infty} \frac{1}{n} \log \frac{w_n}{w_n^*} = \frac{1}{2} \left[ 0.5 \log(4f + 1 - f) + 0.5 \log(0.5f + 1 - f) \right] \tag{24}
   \]

   where ‘\( \log \)’ denotes the binary logarithm. Equation (24) reaches its maximum value with \( f^* = \frac{5}{6} \). This result obviously derives from the limit frequency of booms and drops in our initial construction and could be generalized to other return series as shown in Appendix B.

With \( f^* = \frac{5}{6} \):

\[
\lim_{n \to \infty} \frac{1}{n} \log \frac{w_n}{w_n^*} = \frac{0.25 \log \left( 3 \times \frac{5}{6} \right) + 0.25 \log \left( 1 - 0.5 \times \frac{5}{6} \right)}{2} = 0.25 \log \frac{3.5^2}{6} > 0 \tag{25}
\]

As shown in Figure 3, \( d_n \) outperforms \( r_0, r_1, \ldots, r_n, \ldots \) at infinity. \( \Box \)

The above proposition has an important empirical implication: because of the dominance of geometric strategies over the buy-and-hold alternative, Kelly (1956)’s criterion should be considered as the benchmark to outperform in a trading system allowing partial buying and selling.

Although the rationality of investors practising Kelly strategies is discussed in the literature Samuelson (1963), Aucamp (1993), Benartzi and Thaler (1999), more recent works like Ross (1999), Christensen (2005) affirm that under certain conditions on the utility function, it can also be rational to use geometric strategies and that ‘expected utility theory is rich enough to encompass acceptance as well as rejection of sequences of good bets’ (Ross (1999, p. 325)).

4. Conclusion

This paper develops a computational definition of financial randomness in formulating the concept of ‘unbeatable price sequences’. This latter is constructed on the model of the accepted definition of an infinite random string proposed by Martin-Löf (1966) and Downey and Hirschfeldt (2010).

The concept of outperforming the market is introduced in a renewed framework using theoretical computer science notions that go beyond the traditional probabilistic approach. In the latter, under the EMH and in a one security and cash system, no one can outperform the buy-and-hold in the long run. Our approach puts forward a more general definition of an unbeatable price sequence. For that purpose, we use computable functions to describe effective strategies that are run on price sequences. We then compare their performance to the buy-and-hold benchmark and ultimately decide whether these price sequences are unbeatable or not.

On the technical side, this paper makes four innovations:

1. We introduce a computability definition of randomness that opens new perspectives beyond the traditional probabilistic approach.‡ Statistical conjectures based on samples are no longer necessary in this new framework, and financial price sequences can be studied directly.

2. We use asymptotic limit gains, instead of expected returns, to compare different trading rules. This choice avoids positing the existence of expected returns and thus has a more general implication.

3. Evaluation of financial risks frequently triggers theoretical debates in the literature. Our definition bypassed

‡Zenil and Delahaye (2011) propose a general investigation on this subject.
this difficulty because in the one asset and cash framework that we adopt, the buy-and-hold is the most risky trading rule.

(4) We also distinguish two trading systems: (1) the all-or-nothing trading model; and (2) a more general one that allows partial buying and selling. In this way, we clarify the relation between EMH, the buy-and-hold and the Kelly criterion (Kelly 1956). More precisely, in the all-or-nothing framework, one cannot outperform the buy-and-hold alternative if financial dynamics are i.i.d. variables. However, on allowing partial buying and selling, it is the Kelly’s criterion that should be considered as unbeatable under the EMH.

(5) Consequently, we establish a separation between two concepts (1) unbeatable sequences and (2) unpredictable sequences that, to the best of our knowledge, are not mentioned in Finance and even if to do so, we borrow from Ross (1999) a key argument based on geometric strategies. Using this, we show that one can actually beat buy-and-hold without accurately predicting future drops.

However, one possible limit of our proposition comes from the fact that our reasoning holds at infinity, while financial data are always finite sequences. Consequently, empirical implementations of this paper could remain in an indirect way. For example, to decide if a given strategy outperforms the buy-and-hold alternative, one could choose arbitrarily long time series and observe the global trend followed by their wealth difference.

Another limit of this research is the absence of transaction costs which are important for strategy comparisons. Primb and Yamada (2008). The impacts of this simplification should be analysed accurately to assess the limits of our definition and potential extensions towards formulations explicitly including these transaction costs.

References

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Appendix A. Martin-Löf randomness in a martingale perspective

Among all infinite binary strings, which can be considered as random? In computability theory, this question has been answered in three different ways. According to Martin-Löf (1966), a binary string is random if no effective statistical test can detect any regularity. Chaitin (1969) pledged that the best compression rate one can get from a random string should always tend to zero. Schnorr (1971) considered a binary string as random, if no effective betting system (or effective martingale) can make money on guessing the order of its terms infinitely.

Schnorr (1971) demonstrated the equivalence between these three definitions. And since then, random strings in this sense, named Martin-Löf strings,† serve as a classical reference of randomness in computer science.

Here, we present Martin-Löf strings in a martingale perspective because effective betting systems (in this sense) share some common properties with financial trading rules.

As Downey and Hirschfeldt (2010) recalled:

**Definition Lévy (1937) Appendix A.1.** A martingale is a function \( f : 2^{<\omega} \rightarrow [0, \infty] \) such that for all \( \sigma \):

\[
f(\sigma) = \frac{f(\sigma 1) + f(\sigma 0)}{2}.
\]

It is a supermartingale if for all \( \sigma \),

\[
f(\sigma) \geq \frac{f(\sigma 1) + f(\sigma 0)}{2}.
\]

We say that a (super) martingale succeeds on a string \( \sigma \), if

\[
\limsup_{n \to \infty} f(\sigma \uparrow n) = \infty.
\]

Here, \( 2^{<\omega} \) is the set of all binary strings, \( \sigma \) a particular binary string, \( \mathbb{R} \) the real set and \( \sigma \uparrow n \) the first \( n \) terms of the binary string \( \sigma \).

†Their definition was first proposed by Martin-Löf

![Figure A.1. Profits generated by the martingale \( f(\sigma \uparrow n) \). As one can see in this figure, \( \lim_{n \to \infty} f(r_0, r_1, \ldots, r_n) = \infty \).](image)

**Definition of effective martingale: Appendix A.2.** A (super) martingale \( f \) is said to be effective or computably enumerable (hereafter c.e.) if \( f(\sigma \uparrow n) \) is a c.e. real, and at every stage we have effective approximations to \( f \) in the sense that \( f(\sigma \uparrow n) = \lim_{n \to \infty} f(\sigma \uparrow n) \), with \( f(\sigma \uparrow n) \) a computable increasing sequence of rationals.

The operation of an effective martingale on a binary string can be illustrated with the following example.

Let \( \sigma = r_0, r_2, \ldots, r_n, \ldots \) be a binary string satisfying \( r_{10m} = 0 \) for \( m \in \mathbb{N}^+ \). The effective martingale, \( f(\sigma \uparrow n) \), that bets on 1 at each \( (10m – 1) \)th instant will succeed on \( \sigma \).

Without loss of generality, suppose that \( f(\sigma \uparrow n) \) begins with a capital of 10 monetary units (hereafter m.u.), and makes a 2 m.u. yield (loss) at each time it wins (loses) a bet. As \( f(\sigma \uparrow n) \) does nothing during the first nine steps, we have:

\[
f(r_0) = f(r_0, r_1) = \ldots = f(r_0, r_1, \ldots, r_9) = 10 \quad (A.4)
\]

At the 10th instant, \( f(\sigma \uparrow 10) \) bets on 0. As \( r_{10} = 0 \), the martingale wins 2 m.u. and \( f(r_0, r_1, \ldots, r_9, r_{10}) = 12 \). \( f(\sigma \uparrow n) \) does nothing between the 11th and the 19th steps, and bets on 0 at the 20th stage...

As shown in figure A.1, the effective martingale \( f(\sigma \uparrow n) \) tends to \( \infty \) as \( n \) increases. It can beat the sequence \( \sigma \) in the sense of Schnorr (1971).

Based on the notion of effective martingales, Schnorr (1971) demonstrated that

**Definition Appendix A.3.** An infinite string \( \sigma \) is Martin-Löf random iff no effective (super) martingale succeeds on it.

So, a binary string is random if no effective betting strategy can predict its terms. Our definition of unbeatble sequences is much inspired by this formal relation between randomness and betting systems.

As Martin-Löf strings are used in many propositions in this paper, their properties are recalled in the following:

With a probability of 100%, a Martin-Löf string satisfies all properties of independent observations of an i.i.d. \( U(0, 1) \) random variable.‡

Let \( \sigma \) be a Martin-Löf string, then,

‡In this paper, a random variable \( X \) is i.i.d. \( U(0, 1) \) if \( X \in [0, 1] \), \( \text{Prob}(X = 1) = 0.5 \) et \( \text{Prob}(X = 0) = 0.5 \).
Figure B.1. In this ‘heads or tails’ game, we have 50% chance of making a 60% profit, and 50% chance of a 40% loss.

(1) All effectively extracted infinite subsequences of $\sigma$ remain Martin-Löf strings.
(2) The limit frequency of the term 1 in $\sigma$ is 0.5.
(3) Let $r_n$ be the $n$th term of $\sigma$, then $\sigma$ satisfies:

$$\lim sup_{n \to \infty} \left( \inf \sum_{i=0}^{n} r_i - \frac{1 \times n + 0 \times n}{2} \right) = \sqrt{n} \left( \frac{-\sqrt{n}}{2} \right) = \infty (-\infty).$$

(A.5)

Appendix B. Geometric strategies

Geometric strategies are first proposed by Bernoulli (1738) to answer the question ‘How much should I bet in repetitive games?’ According to Bernoulli (1738), in this kind of situation, one should always maximize the geometric mean of his gains.

For example, the geometric mean $E(\log(X))$ of the following ‘heads or tails’ game can be calculated by equation (B.1).

$$E(\log(X)) = 50\% \log(160\% f + (1 - f)) + 50\% \log(60\% f + (1 - f))$$

(B.1)

Here, $E(.)$ means the expectation function, $f$ the percentage to bet. On maximizing $E(\log(X))$, the percentage to be invested $f^* = \frac{5}{12}$. So, the trading rule betting $\frac{5}{12}$ of its wealth at each game is called ‘geometric strategy’ or Kelly strategy Kelly (1956).

In finance, there is controversy about the rationality of geometric strategy. Samuelson (1963) found it irrational to execute a ‘geometric strategy’, as the loss (of 40% in our example) associated with the pessimist option would prevent rational investors accepting the game. Behaviourists Benartzi and Thaler (1999) explained Samuelson’s conclusion by psychological biases. However, Ross (1999) remarked that, under certain conditions on the utility function, it can also be rational to accept geometric strategies and that ‘expected utility theory is rich enough to encompass acceptance as well as rejection of sequences of good bets’. More recently, there seems to be a consensus on the role of geometric strategy in portfolio management Christensen (2005).

While this theoretical discussion is beyond our subject, geometric strategies have an important impact on the definition of unbeatable sequences: buy-and-hold is unbeatable only in buying all and selling all systems. In the general case, it can be beaten by geometric strategies even if return series are completely unpredictable.