
THE SET OF PERIODIC POINTS

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Introduction. The aim of this paper is to answer a question raised by T. Y. Li and J. A. Yorke in this MONTHLY [6]: “Is the closure of the periodic points [of a continuous function which maps an interval into itself] an interval or at least a finite union of intervals?”

The notion of “periodic point” may be understood in the strict sense or in a broad one, but in each case the answer is “no.” This is what we show by using general methods for constructing functions with an explicitly known set of periodic points.

After some definitions we give, in Section 2, a general proposition, which is of interest in itself, concerning the set of fixed points (periodic points of period 1). In Section 3 we complete the answer when “periodic point” is understood in the strict sense, and in Section 4 we solve the other case with an appropriate counterexample.

1. Definitions. Let $f$ be a continuous function from $[0, 1]$ to $[0, 1]$ (no generality is lost by choosing $[0, 1]$), and let $f^p(x)$ denote the $p$th iterate of $f$.

We say that $x$ is a periodic point of period $p$ if:

$$f^p(x) = x \quad \text{and} \quad \forall i \in \{1, 2, \ldots, p - 1\} : f^i(x) \neq x.$$ 

We say that $x$ is an ultimately periodic point of period $p$ if there exists some $n \in \mathbb{N}$ such that:

$$f^n(x)$$ is a periodic point of period $p$.

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We say that \( x \in [0, 1] \) is an asymptotically periodic point of period \( p \) if, when we define 
\[
x_0 = x, \ x_{n+1} = f(x_n),
\]
the sequences \( \{x_{np}\}, \{x_{np+1}\}, \ldots, \{x_{np+p-1}\} \) converge to \( p \) different points.

A periodic point is an ultimately periodic point; an ultimately periodic point is an asymptotically periodic point; and a point which is not an asymptotically periodic point is called a turbulent point, because in this case the sequence \( \{x_n\} \) possesses an infinite set of accumulation points.

For sets of accumulation points of such sequences see [2], [4], [5], [7], [8], [10], and for periodic points see [1], [3], [6], [9].

2. The Set of Fixed Points.

**Proposition.** 1. (i) The set of fixed points of a continuous function from \([0, 1]\) to \([0, 1]\) is a closed subset of \([0, 1]\).

(ii) For every closed subset \( F \) of \([0, 1]\) there exists a continuous function \( f \) from \([0, 1]\) to \([0, 1]\) whose fixed point set is \( F \).

**Proof.** (i) Trivial.

(ii) Let \( F \) be a closed subset of \([0, 1]\). We assume that \( 0 \in F \) and \( 1 \in F \) (if this is not the case, it is easy to modify the proof).

The set \([0, 1] - F = U\) is an open set, and consequently \( U \) is a denumerable union of disjoint open intervals \((a_i, b_i), i \in \mathbb{N}\). (We only consider the case of the infinite union.)

We define a sequence \( \{f_n\} \) of continuous functions:

\[
\forall n \in \mathbb{N}, \begin{cases} 
\forall x \in F \cup \left( \bigcup_{i=n}^{\infty} (a_i, b_i) \right): f_n(x) = x \\
\forall i < n \begin{cases} 
\forall x \in [a_i, (a_i + b_i)/2]: f_n(x) = a_i \\
\forall x \in [(a_i + b_i)/2, b_i]: f_n(x) = 2x - b_i .
\end{cases}
\end{cases}
\]

The sequence \( \{f_n\} \) converges uniformly to a function \( f \). Therefore \( f \) is continuous and one can see that its fixed point set is exactly \( F \) and that there is no other periodic point.

In Fig. 1 we show the continuous function whose fixed point set is the Cantor triadic set (i.e., the set of points of \([0, 1]\] having a development in base \(3\) that contains only 0 and \(2\)).

**Remarks.** 1. If \( F \) is a closed set which is not a finite or denumerable union of intervals (for example, the Cantor set) the function \( f \), above, answers the question of Li and Yorke.

2. Proposition 1 is still true when one replaces \([0, 1]\] by \( \mathbb{R} \) or \( \mathbb{R}^n \).

3. One can also easily improve part (i) of Proposition 1 in another direction: Let \( f \) be a continuous function from \([0, 1]\) to \([0, 1]\] and let \( n \) be an integer different from 0. The set of periodic points of period \( p \) such that \( p \mid n \) (or \( p \leq n \)) is a closed subset of \([0, 1]\].

But the set of all periodic points is not necessarily closed. For example, consider the continuous function \( h \) defined as follows:

\[
\forall x \in [0, 1/2]: h(x) = 2x; \forall x \in [1/2, 1]: h(x) = 2 - 2x .
\]

One can establish (using binary developments) that the set of periodic points of \( h \) is a denumerable dense subset of \([0, 1]\].


3. The Set of Periodic Points. We shall say that a subset \( F \) of \([0, 1]\) is symmetric if:

\[
\forall x \in [0, 1]: 1/2 + x \in F \iff 1/2 - x \in F .
\]

**Proposition.** 2. (i) The set of periodic points of period 1 or 2 of a continuous function from \([0, 1]\) to \([0, 1]\] is a closed subset of \([0, 1]\].

(ii) For every symmetric closed subset of \([0, 1]\) there exists a continuous function from \([0, 1]\) to \([0, 1]\] whose set of periodic points of period 1 or 2 is \( F \cup \{1/2\} \).
Proof. (i) Trivial.
(ii) Let $F$ be a symmetric closed subset of $[0, 1]$. We assume that $0 \in F$ and $1/2 \in F$ (if this is not the case it is easy to modify the proof).

The set $[0, 1/2] - F \cap [0, 1/2] = U$ is open, and consequently $U$ is a denumerable union of disjoint open intervals $(a_i, b_i)$, $i \in \mathbb{N}$. (We consider only the case of an infinite union.)

We define a sequence $\{l_n\}$ of continuous functions:

$$
\forall n \in \mathbb{N} \left\{ \begin{array}{ll}
\forall x \in [1/2, 1] \cup F \cup \left( \bigcup_{i=n}^{\infty} (a_i, b_i) \right) : & l_n(x) = -x \\
\forall i < n \left( \begin{array}{ll}
\forall x \in (a_i, (a_i + b_i)/2) : & l_n(x) = 1 - 2x + a_i \\
\forall x \in ((a_i + b_i)/2, b_i) : & l_n(x) = 1 - b_i.
\end{array} \right)
\end{array} \right.
$$

The sequence $\{l_n\}$ converges to a function $l$, uniformly. Therefore $l$ is continuous and one can see that:

- $l$ has the unique fixed point $1/2$;
- the set of periodic points of period 2 is $F - \{1/2\}$;
- there is no other periodic point.
REMARKS. 1. Since the Cantor set is symmetric, the function \( l \) gives us another answer to the question of Li and Yorke.

2. By analogous methods one can obtain continuous functions having only periodic points of period \( 2^i, 2^{i-1}, \ldots, 2, 1 \) and such that the periodic point set is a very irregular closed subset of \([0, 1]\) (i.e., not a denumerable union of intervals).

4. The Set of Ultimately or Asymptotically Periodic Points. If we understand “periodic point” in a generalized sense (i.e., ultimately periodic point or asymptotically periodic point) the answer is still the same, but it is obtained in a very different way.

Let \( g \) be a function from \([0, 1]\) to \([0, 1]\). We consider the following properties:

\[
g \text{ is continuous}
\forall i \in \mathbb{N} \left\{
\forall x \in [1 - 1/3^i, 1 - 2/3^{i+1}]: \ g(x) = x - 1 + 5/3^{i+1}
\right.
\text{g decreases on } [1 - 2/3^{i+1}, 1 - 1/3^{i+1}].
\]

(\#)

PROPOSITION 3. If \( g \) satisfies (\#), then for each \( j \in \mathbb{N} \):

(i) \( 2/3^{j+1} \) is a turbulent point,
(ii) there exists a unique solution \( y_j \) of the equation
\[
g(x) = x - (3^j - 1)/3^j;
\]
this point \( y_j \) is a periodic point of period \( 2^j \) and \( y_j \in \left[ 1 - 2/3^{j+1}, 1 - 1/3^{j+1} \right] \).

(The proof depends on a development in base 3.)

**Remark.** Functions verifying \((\ast)\) exist and have been used to solve another question about periodic points [3]. We now can define our last counterexample \( g \) (see Fig. 2):

\[
\begin{align*}
g(1) &= 0 \\
\forall i \in \mathbb{N} \quad &\begin{cases} 
\forall x \in [1 - 1/3^i, 1 - 2/3^{i+1}] : & g(x) = x - 1 + 5/3^{i+1} \\
\forall x \in [1 - 2/3^{i+1}, 1 - 5/3^{i+2}] : & g(x) = 3 - 3x - 1/3^i \\
\forall x \in [1 - 5/3^{i+2}, 1 - 4/3^{i+2}] : & g(x) = 2/3^{i+1} \\
\forall x \in [1 - 4/3^{i+2}, 1 - 1/3^{i+1}] : & g(x) = 4 - 4x - 10/3^{i+2}.
\end{cases}
\end{align*}
\]

The function \( g \) satisfies \((\ast)\); from the definition and proposition 3(i), it follows that if \( x \in X = \bigcup_{i=0}^{\infty} [1 - 5/3^{i+2}, 1 - 4/3^{i+2}] \) then \( x \) is a turbulent point.

We have:
\[
[0, 1] - X = \left[ 0, \frac{4}{9} \right] \cup \left( \bigcup_{i=0}^{\infty} \left( 1 - 4/3^{i+2}, 1 - 5/3^{i+3} \right) \right) \cup \{ 1 \}.
\]

For every \( i \in \mathbb{N} \), \( y_j \in (1 - 4/3^{i+2}, 1 - 1/3^{i+1}) \). Therefore there is at least one periodic point in every interval \((1 - 4/3^{i+2}, 1 - 5/3^{i+3})\) (and consequently at least one ultimately periodic point and at least one asymptotically periodic point).

If we denote by \( U \) the set of ultimately periodic points; by \( A \), the set of asymptotically periodic points; and by \( \overline{U}, \overline{A} \), their closures, we have:
\[
\overline{U} \subset \left[ 0, \frac{4}{9} \right] \cup \left( \bigcup_{i=0}^{\infty} \left[ 1 - 4/3^{i+2}, 1 - 5/3^{i+3} \right] \right) \cup \{ 1 \}
\]
\[
\overline{A} \subset \left[ 0, \frac{4}{9} \right] \cup \left( \bigcup_{i=0}^{\infty} \left[ 1 - 4/3^{i+2}, 1 - 5/3^{i+3} \right] \right) \cup \{ 1 \}
\]
\[
\forall i \in \mathbb{N} \quad \overline{U} \cap \left[ 1 - 4/3^{i+2}, 1 - 5/3^{i+3} \right] \neq \emptyset
\]
\[
\overline{A} \cap \left[ 1 - 4/3^{i+2}, 1 - 5/3^{i+3} \right] \neq \emptyset.
\]

This implies that \( \overline{U} \) and \( \overline{A} \) are not finite unions of intervals.

**Remarks.** 1. Is \( \overline{U} \) or \( \overline{A} \) a denumerable union of intervals? Even in this case (i.e., for \( g \)) the answer is unknown.

2. Like the set of periodic points, \( U \) and \( A \) are not closed in general. For example, for the function \( h \) (see Section 3) we have the surprising result that \( U = A = Q \cap [0, 1] \).

**References**


**ON CAUSTICS OF PLANE CURVES**

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When light from a small source is reflected from the rounded inner surface of a teacup and scattered from the surface of the tea (which ideally should contain milk!) we see a bright curve on the surface. The curve will have cusps or double points which move and change when the cup is tilted.

The bright curve, or caustic, is caused by a concentration of light rays along the envelope of the reflected rays, and it only becomes visible when a screen (in this case the surface of the tea) scatters the light to our eyes.

Caustics have been studied for about 300 years, from the time of Huygens [8]. Old books on geometrical optics, such as [6] and [7], have something to say about them, and they appear in some books about curves, such as [9] and [11]. Cayley [2] wrote a memoir in 1857 in which he considered not only reflection but also refraction, and gave detailed calculations in the case of a circle.

In this paper we describe a simple geometrical technique, based on conics, for obtaining properties of caustics in the simplest case of a "mirror" which is a smooth curve in the plane, the light source also being in this plane. By this technique, various general properties, as well as special examples, can be studied directly from the mirror and without long special calculations. In

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J. W. Bruce wrote his Ph.D. thesis under the supervision of C. T. C. Wall at Liverpool University. After a year at I.H.E.S., Bures-sur-Yvette, and a year as a postdoctoral fellow at Liverpool, he is now on the staff of University College, Cork, Ireland. His research interests are in singularity theory, particularly its applications to algebraic and differential geometry.

P. J. Giblin and C. G. Gibson are on the staff at Liverpool University, where they work with other members of the department in singularity theory. They have collaborated with Bruce on an extensive investigation of caustics in two or more dimensions, of which this article is the geometrical beginning. Giblin did his postgraduate work under J. E. Reeve at King's College, London; and Gibson, who was at the time primarily interested in intuitionism, under A. Heyting in Amsterdam.—Editors