Connectivity Queries on Curves in $\mathbb{R}^n$

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Canny introduced the notion of *roadmap* in [2], as a way to study connectivity properties of semi-algebraic sets (which appear for instance in motion planning problems).

A roadmap R of a semi-algebraic set V is a curve contained in V, that has a non-empty and connected intersection with each connected component of S. Given two query points A, B on V, it is possible to construct a roadmap that contains both of them. Then, A and B belong to the same connected component of V if and only if they are on the same connected component of R. Thus, roadmaps allow one to reduce connectivity queries on semi-algebraic sets to connectivity queries on curves.

Let n be the dimension of the ambient space, and let $X_1, \dots, X_n$ be coordinates in $\mathbb{C}^n$. Following Canny's algorithm, and improvements by Basu, Pollack and Roy [1], the roadmap algorithm from [5] computes the following:

1. two linear forms $\eta = \eta_1 X_1 + \cdots + \eta_n X_n$ and $\vartheta = \vartheta_1 X_1 + \cdots + \vartheta_n X_n$, with coefficients in $\mathbb{Q}$
2. polynomials $q, q_0, \dots, q_n$ in $\mathbb{Q}[T, U]$ where T and U are indeterminates.

Let $Z \subset \mathbb{C}^n$ be the constructible set defined by

$$q(\eta, \tau) = 0, \qquad X_i = \frac{q_i(\eta, \tau)}{q_0(\eta, \tau)} \ \ (1 \le i \le n), \qquad q_0(\eta, \tau) \neq 0.$$

Then, the roadmap R is obtained as $C \cap \mathbb{R}^n$, where $C \subset \mathbb{C}^n$ is the algebraic curve obtained as the Zariski closure of Z.

In this work, we consider this roadmap as our input. Given $(q, q_0, \dots, q_n)$ and η, ϑ, as well as two query points A, B on R, our question is to decide whether A and B are on the same connected component of R. To our knowledge, no previous work directly addresses this question. A close reference is in [6], which however considers a more general input (given by means of a regular chain), and relies on Puiseux series computations.

The algorithm we propose is inspired by El Kahoui's algorithm for the topology of a space curve [4]; we also use ideas from [7, 3], that allow us to replace computations with real algebraic numbers by manipulations on isolating boxes.

Our algorithm, as well as in El Kahoui's, requires that the input curve be in general position. The genericity requirements are of a geometric nature (e.g., there should be no point on R with a tangent orthogonal to the η, ϑ-plane, etc).

Of course, these conditions can be ensured by means of a generic enough change of coordinates $\mathbf{A}$; we can also suppose that the linear forms η, ϑ are X_1, X_2. We give a precise cost estimate for the application of this change of coordinates; we also prove that the set of all unlucky $\mathbf{A}$ is contained in a strict algebraic subset of GL_n of degree $\delta^{O(1)}$, where δ is the degree of C. Using Zippel-Schwartz's lemma, this allows us to determine the probability of success of finding a generic enough change of coordinates $\mathbf{A}$ in a large finite subset of GL_n.

Supposing that the chosen change of coordinates is generic, our algorithm works in three steps:

1. one computes a rational parametrization $(q', q'_0, \ldots, q'_n), X_1, X_2$ of $C^{\mathbf{A}} = \{\mathbf{A}z \mid z \in C\}$ using fast algorithms for change of orderings in triangular sets;
2. one computes the topology of the plane curve defined by $q'(X_1, X_2) = 0$ using e.g. [3];
3. we consider the space curve defined as the Zariski-closure of the constrictible set defined by

$$q'(X_1, X_2) = 0, \quad X_3 = \frac{q'_3(X_1, X_2)}{q'_0(X_1, X_2)}, \quad q_0(X_1, X_2) \neq 0.$$

   Using results from [4], we deduce the topology of the space curve from the one computed in Step 1, and use it to answer connectivity queries on the space curve.

The algorithm works because genericity properties of $\mathbf{A}$ allow us to prove that connected components of the curve $C^{\mathbf{A}} \subset \mathbb{R}^n$ are in one-to-one correspondance with the connected components of its projection on the (X_1, X_2, X_3)-space.

References

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