Computer algebra methods for the stability analysis of differential systems with commensurate time-delays

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Abstract—This paper is devoted to the study of the stability of linear differential systems with commensurate delays. Within the frequency-domain approach, it is well-known that the asymptotic stability of such systems is ensured by the condition that all the roots of the corresponding quasipolynomial have negative real parts. A classical approach for checking this condition (see \([1]\)) consists in computing the set of critical zeros of the quasipolynomial, \(K\), the roots (and the corresponding delays) of the quasipolynomial that lie on the imaginary axis, and then analyzing the variation of these roots with respect to the variation of the delay. Following this approach, based on solving algebraic systems techniques, we propose a certified and efficient symbolic-numeric algorithm for computing the set of critical roots of a quasipolynomial. Moreover, using recent algorithmic results developed by the computer algebra community, we present an efficient algorithm for the computation of Puiseux series at a critical zero which allows us to finely analyze the stability of the system with respect to the variation of the delay \([2]\). Explicit examples are given in order to illustrate our algorithms.

I. INTRODUCTION

This paper aims at studying the stability of the class of linear time-invariant (LTI) differential systems with commensurate time-delays by means of computer algebra methods recently developed by the symbolic computation community.

An example of such a system of neutral-type is defined by the following state-space representation

\[
\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - k \tau) + \sum_{l=1}^{m} B_l \dot{x}(t - l \tau),
\]

where \(\tau \in \mathbb{R}^+ := \{ \tau \in \mathbb{R} | \tau \geq 0 \}\), \(m \in \mathbb{N}^+ := \{0, 1, \ldots\}\), \(A_0, \ldots, A_m\), and \(B_1, \ldots, B_m\) are constant matrices with entries in a field \(\mathbb{K}\) (e.g., \(\mathbb{K} = \mathbb{Q}, \mathbb{R}\)). The characteristic function of (1) is defined by a quasi-polynomial of the form

\[
f(s, \tau) = \sum_{j=0}^{m} p_j(s) e^{-j \tau s},
\]

where the \(p_j\)'s are polynomials in the complex variable \(s\) with coefficients in \(\mathbb{K}\).

In this paper, we investigate the asymptotic stability of LTI differential commensurate time-delay systems whose dynamics are defined by quasipolynomials of the form of (2). Recall that such a system is said to be asymptotically stable if all the zeros of the quasi-polynomial \(f(s, \tau)\) have negative real parts, i.e., \(f(s, \tau) \neq 0\) for all \(s\) in the closed right half-plane \(\mathbb{C}_+ := \{ s \in \mathbb{C} | \Re(s) \geq 0 \}\). Our approach for analyzing the asymptotic stability of this class of systems is based on the so-called critical pairs of \(f(s, \tau)\), that is the pairs \((\omega \in \mathbb{R}, \tau \in \mathbb{R}^+)\) such that \(f(i \omega, \tau) = 0\). For instance, see, e.g., \([1]\), \([3]\), \([4]\), \([5]\), \([6]\) and the references therein. If such critical pairs exist, the stability is derived from the asymptotic behavior of the coordinates \(s\) of these pairs, called critical imaginary roots of \(f(s, \tau)\), that is the way these critical imaginary roots behave under small variation of the time-delay \(\tau\). There are two independent steps to analyze the stability. First, the critical pairs of \(f(s, \tau)\) are computed and then for each critical pair \((s_0, \tau_0)\), the asymptotic behavior of the root \(s_0\) with respect to small variation of \(\tau_0\) has to be studied. For more details, see, e.g., \([1]\), \([3]\), \([4]\), \([5]\), \([6]\) and the references therein.

There exist several methods for computing the critical pairs of the system. For the study of the asymptotic behavior of critical imaginary roots, several methods have been developed for the case of simple imaginary roots (see, e.g., \([1]\), \([3]\), \([4]\), \([5]\), \([6]\) and the references therein). For multiple imaginary roots (MIR), a recent method based on Puiseux series was developed in \([1]\), \([2]\).

The contributions of this paper are twofold. We first present a new approach for the computation of the critical pairs. Applying some transformations, this approach roughly reduces the computation of the critical pairs of a quasipolynomial to the computation of the real solutions of a zero-dimensional polynomial system in two variables, i.e., a system admitting a finite number of complex solutions. This can be achieved using the so-called Rational Univariate Representation (RUR) \([7]\) which is, roughly speaking, a one-to-one mapping between the solution of the polynomial system and the roots of a univariate polynomial.

Our second contribution concerns the study of the asymptotic behavior of the quasipolynomial \(f(s, \tau)\) at a critical pair \((s_0, \tau_0)\). More precisely, using recent advances obtained in the computer algebra community, we present an efficient algorithm that allows us to compute the different terms of Puiseux series. Hence, we can study the variation \(\Delta s\) in terms of \(\Delta \tau\) in a neighborhood of a critical pair \((s_0, \tau_0)\).
The paper is organized as follows. In Section II, we present two efficient and certified algorithms which compute the critical pairs. An algorithm which computes Puiseux series expansions is given in Section III.

II. COMPUTATION OF THE CRITICAL PAIRS

In this section, we focus on the certified computation of the critical pairs of the quasipolynomial \( f(s) \), that is its zeros of the form \((i\omega, \tau)\), where \((\omega, \tau) \in \mathbb{R} \times \mathbb{R}_+\). This problem is equivalent to computing the set of real zeros of \( f(i\omega, \tau) \). Due to the presence of transcendental terms, this quasipolynomial usually admits an infinite number of zeros. However, by means of some particular transformations, this problem can be reduced to the computation of the real solutions of a zero-dimensional polynomial system, i.e., a polynomial system admitting only a finite number of complex solutions. In what follows, we present two kinds of transformations. The first one, the Rekasius transformation, appears in [8] and was used in a series of works [5], [6]. The second one is a Möbius transformation.

Rekasius transformation. This transformation consists in replacing the terms \( e^{-\tau i\omega} \) in the quasipolynomial \( f(i\omega, \tau) \) by the rational fraction \( \frac{1}{1 + T^2 i\omega} \), where \( T \in \mathbb{R} \). Cleaning the denominators, we obtain a polynomial of the form \( R(\omega, T) + iI(\omega, T) \). Computing the critical pairs then amounts to studying the real zeros of the following polynomial system:

\[
\begin{align*}
R(\omega, T) &= 0, \\
I(\omega, T) &= 0.
\end{align*}
\]  

(3)

Given a solution \((\omega, T) \in \mathbb{R}^2\) of (3), the critical delays can then be obtained by:

\[
\tau_k = \frac{2}{\omega} \left( \arctan(\omega T) + k\pi \right), \quad k \in \mathbb{Z}.
\]  

(4)

Möbius transformation. An alternative to Rekasius transformation consists in replacing the terms \( e^{-\tau i\omega} \) by a new variable \( z \). When the delay \( \tau \) varies, \( e^{-\tau i\omega} \) covers the complex torus \( \mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \} \). The problem of studying the zeros of \( f(i\omega, \tau) \) then amounts to studying the zeros of the polynomial \( f(i\omega, z) \), where \( \omega \in \mathbb{R} \) and \( z \in \mathbb{T} \). For more details, see, e.g., [1] and the references therein.

A first approach is to consider \( z = u + i v \), where \( u, v \in \mathbb{R} \), so that \( f(i\omega, u + i v) = R(\omega, u, v) + iI(\omega, u, v) \), where \( R \) and \( I \) are two polynomials with real coefficients. The problem then leads to the problem of computing the real solutions \((\omega, u, v)\) of the following polynomial system:

\[
\begin{align*}
R(\omega, u, v) &= 0, \\
I(\omega, u, v) &= 0, \\
u^2 + v^2 - 1 &= 0.
\end{align*}
\]

Generically, the above system is zero-dimensional, i.e., it only admits a finite number of complex solutions. Standard computer algebra methods such that the Rational Univariate Representation (RUR) [7] can then be used to obtain certified numerical approximations of the real solutions.

In order to reduce the number of indeterminates in the above equivalent polynomial system, we propose the Möbius transform \( \frac{z}{z+1} \) which maps \( \mathbb{T} \) to \( \mathbb{R} \cup \{ \infty \} \). Similarly as above, substituting \( e^{-\tau i\omega} \) by \( \frac{1}{z+1} \) into \( f(i\omega, \tau) \) and cleaning the denominators, we obtain a polynomial of the form \( R(\omega, x) + iI(\omega, x) \), where \( R \) and \( I \) are two polynomials with real coefficients. The critical pairs can then be computed by solving the following polynomial system

\[
\begin{align*}
R(\omega, x) &= 0, \\
I(\omega, x) &= 0,
\end{align*}
\]  

(5)

and then the critical delays are obtained as follows:

\[
\tau_k = \frac{1}{\omega} \left( \arctan \left( \frac{x - 1}{x + 1} \right) + 2k\pi \right), \quad k \in \mathbb{Z}.
\]  

(6)

Remark 1: We can observe that \( z = 1 \) is sent to \( \infty \) in the above Möbius transformation. This case, which corresponds to the delays of the form \( 2k\pi \), where \( k \in \mathbb{Z} \), can be independently studied. The corresponding quasipolynomial is then a standard polynomial in \( \omega \) whose real roots can be isolated using, e.g., classical bisection algorithms [9], [10].

Remark 2: Notice that, up to a sign, the Rekasius transformation is equivalent to the Möbius transformation by setting \( x = -T\omega \).

For the above transformations, the computation of the critical pairs can be reduced to solving (3) or (5).

In what follows, we only consider the case of the Möbius transformation (the approach being similar for Rekasius transformation). Let us now consider the polynomial system (5) resulting from the above transformation. Without loss of generality, we can assume that (5) admits only a finite number of complex solutions, which means that the polynomials \( R \) and \( I \) do not have a non-trivial common factor. Our objective is to study the solutions of (5) by means of modern computer algebra methods.

A first important remark is that the \( \omega \)-coordinates of the solutions of \( f(i\omega, \tau) = 0 \) (called the imaginary roots of \( f \), are actually the roots of a resultant computed from \( R \) and \( I \). More precisely, given two polynomials \( f = \sum_{i=0}^{n} a_i x^i \) and \( g = \sum_{i=0}^{m} b_i y^i \), the resultant of \( f \) and \( g \), denoted by \( \text{Res}(f, g) \), is the determinant of the Sylvester matrix (see Appendix).

The resultant \( \text{Res}_x(R, I) \) of \( R, I \in \mathbb{Q}[\omega, x] \), seen as univariate polynomials in \( x \) with coefficients in the ring \( \mathbb{Q}[\omega] \), has the following property.

Theorem 1: The roots of \( \text{Res}_x(R, I) \) are the projection onto the \( \omega \)-axis of the common solutions of \( R \) and \( I \) and the common roots of \( a_n \) and \( b_m \).

If we want to check whether or not (5) admits imaginary roots, we can first compute \( \text{Res}_x(R, I) \) and then check whether or not \( \text{Res}_x(R, I) \) admits real roots. A method to achieve this is, for instance, to use the Descartes rule of sign [10]. If \( f(i\omega, \tau) \) does not admit solutions with real \( \omega \)-coordinates, then it means that the stability of the system...
is independent from the delay $\tau$. In that case, a method for checking the stability consists in substituting, for instance, $\tau = 0$ in $f(s, \tau)$ and then applying the standard Routh criterion to the univariate polynomial $f(s, 0)$.

From the end-user point of view, a convenient way to express the solutions of (5) is to use a RUR [7], that is a one-to-one mapping

$$V(\langle R, I \rangle) \rightarrow V(\langle f_h \rangle)$$

$$\begin{pmatrix} x \\ \omega \end{pmatrix} \mapsto \begin{pmatrix} g_h(x) \\ g_h(\omega) \end{pmatrix},$$

(7)

between the solutions of (5) defined by

$$V(\langle R, I \rangle) := \{(\omega, x) \in \mathbb{K}^2 \mid R(\omega, x) = \mathcal{I}(\omega, x) = 0\},$$

and the roots $V(\langle f_h \rangle) := \{t \in k \mid f_h(t) = 0\}$ of a certain univariate polynomial $f_h$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In order to achieve the one-to-one condition, the representation (7) is computed with respect to a separating linear form $h := a_1 \omega + a_2 x \in \mathbb{Q}[\omega, x]$, for certain $a_1, a_2 \in \mathbb{Q}$, that takes different values when evaluated at the different points of $V(\langle R, I \rangle)$. Using (7), the solutions of $V(\langle R, I \rangle)$ are then of the following form

$$\begin{cases} f_h(t) = 0, \\ \omega = g_h(\omega)(t), \\ x = g_h(x)(t), \end{cases}$$

(8)

where $f_h, g_h, g_h, g_h, g_h, g_h, x \in \mathbb{Q}[t]$ and $f_h$ and $g_h$ are coprime polynomials, i.e., $\gcd(f_h, g_h) = 1$.

Computing a RUR requires solving the following two distinct problems:

- Find a separating linear form $h := a_1 \omega + a_2 x \in \mathbb{Q}[\omega, x]$.
- Given a linear form $h \in \mathbb{Q}[\omega, x]$, compute a RUR-candidate, that is the polynomials $f_h, g_h, g_h, g_h, g_h, g_h, g_h, g_h, x \in \mathbb{Q}[t]$ and $f_h$ and $g_h$ are coprime polynomials, i.e., $\gcd(f_h, g_h) = 1$.

**Computation of the RUR-candidate.** Given a polynomial $h$ in $\mathbb{Q}[\omega, x]$, the RUR-candidate with respect to $h$ can be computed using the algorithm given in [7]. This algorithm requires the knowledge of a basis of the finite $\mathbb{Q}$-vector space $\mathcal{A} := \mathbb{Q}[\omega, x] / \langle R, I \rangle$ and a reduction algorithm which computes normal forms modulo the ideal $\langle R, I \rangle$. In order to explicitly characterize the polynomials appearing in the RUR-candidate, we first define the $\mathbb{Q}$-endomorphism defined by the multiplication by $h \in \mathbb{Q}[\omega, x]$ in $\mathcal{A}$ given by

$$m_h : \mathcal{A} \rightarrow \mathcal{A}$$

$$p \mapsto \overline{hp},$$

where $\overline{p}$ denotes the residue class of $p \in \mathbb{Q}[\omega, x]$ in $\mathcal{A}$ (i.e., modulo $\langle R, I \rangle$). A representative of $\overline{p}$ is the normal form of $p$ with respect to the Gröbner basis of $\langle R, I \rangle$.

Given a basis $\{e_1, \ldots, e_n\}$ of $\mathcal{A}$ (which can be deduced from a Gröbner basis for any monomial ordering [11]), we can compute the $(n \times n)$-matrix $M_h$ associated to $m_h$. The polynomial $f_h$ of the RUR-candidate is defined by the characteristic polynomial of the matrix $M_h$. Moreover, if

$$\overline{f_h} := \frac{f_h}{\gcd(f_h, \frac{df_h}{dt})} = \sum_{i=0}^{d} v_i t^{d-i} \in \mathbb{Q}[t]$$

denotes the square-free part of $f_h$, and if we note

$$H_j := \sum_{i=0}^{j} v_i t^{d-i} \in \mathbb{Q}[t], \quad j = 0, \ldots, d - 1,$

then we can define:

$$\begin{align*}
 g_h & := \sum_{i=0}^{d-1} \text{Trace}(M_h^i) H_{d-i-1}, \\
 g_h & := \sum_{i=0}^{d-1} \text{Trace}(M_h^i) H_{d-i-1}, \\
 g_h, x & := \sum_{i=0}^{d-1} \text{Trace}(M_h^i) H_{d-i-1}.
\end{align*}$$

**Finding of a separating linear form.** For the computation of a separating polynomial of $V(\langle R, I \rangle)$, a critical remark is that the number of non separating elements is bounded by $m := n(n - 1)/2$, where $n$ denotes the cardinal of $V(\langle R, I \rangle)$, that is the number of lines passing by two distinct points of $V(\langle R, I \rangle)$. Thus, a separating form can always be found among the set $\{h := \omega + a x \mid a = 0, \ldots, m\}$. On the other hand, for polynomials $R$ and $I$ of total degree $d$, by the Bezout theorem [11], [12], we know that the cardinal of $V(\langle R, I \rangle)$ is bounded by $d^2$. Hence, a strategy for computing a separating element for $V(\langle R, I \rangle)$ is to loop over $d^2$ ($d^2 - 1)/2$ different integers $a$, for each $a$, compute the number of distinct roots of the polynomial $f_h$ (see above), i.e., the degree of its squarefree part $\overline{f_h}$, and finally select an $a$ for which this number is maximal. This ensures that the degree of $\overline{f_h}$ is equal to the cardinal of $V(\langle R, I \rangle)$, and thus that the roots of $f_h$ are in bijection with the points of $V(\langle R, I \rangle)$. In practice, since an arbitrary chosen linear form is separating with high probability, one prefers a strategy that consists in choosing randomly a linear form and testing that the latter is separating a posteriori of the computation of a RUR-candidate. This test is based on the fact that a linear form $h$ is separating for $V(\langle S \rangle)$ if and only if the polynomials $u_\omega := g_h \omega - g_h, \omega$ and $u_x := g_h \omega - g_h, x$ vanish on all the points of $V(\langle R, I \rangle)$ (i.e., belong to the radical of $\langle R, I \rangle$). In computational terms, this condition can be translated into $[\text{Trace}(M_{u_\omega, e_i})]_{i=1, \ldots, n} = [0, \ldots, 0]$ and $[\text{Trace}(M_{u_x, e_i})]_{i=1, \ldots, n} = [0, \ldots, 0]$. For more details, see [7].

**Efficient RUR algorithm.** Actually, the above RUR algorithm is designed for general zero-dimensional systems and is thus not optimized for the specific case of two polynomials in two variables. In particular, the computation of a Gröbner basis of $\langle R, C \rangle$, required for the computation of the RUR-candidate as well as for the separating form, is often time-consuming.
Alternatively, below, we propose a practically efficient method that computes a decomposition into univariate representations of the solutions of $S$ and which avoids the computation of a Gröbner basis. This method consists in shearing the system, i.e., applying a change of variables and then using resultants and subresultants (See Appendix) to compute parametrizations of the solutions. Providing that the change of variables sends the system into a generic position, the computed parametrization then encodes the solutions of the system in a one-to-one correspondence with the roots of a univariate polynomial.

Let us start with the following result which shows that there exist $h := \omega + a x$, up to a factor in $\mathbb{Q}$, the polynomial $f_h$ is equal to the resultant of two polynomials resulting from $R$ and $I$ after a change of variables.

**Theorem 2** ([13]): Let $R(\omega, x)$, $I(\omega, x) \in \mathbb{Q}[\omega, x]$. Define $R'(t, x) := R(t - a x, x)$ and $I'(t, x) := I(t - a x, x)$, where $a \in \mathbb{Z}$ is such that the leading coefficient of $R'$ and $I'$ with respect to $x$ are coprime. Then, the resultant of $R'$ and $I'$ with respect to $x$, denoted $\text{Res}_x(R', I')$, is equal to:

$$c \prod_{(\alpha_1, \alpha_2) \in V((R, I))} (t - \alpha_1 - a \alpha_2)^{\mu(\alpha_1, \alpha_2)}, \quad c \in \mathbb{Q}.$$ 

Given a linear form $\omega + a x$, it is well-known that the latter is separating for $V((R, I))$ if and only if the system $(R', I')$ is in generic position, i.e., for each root $\alpha$ of $\text{Res}_x(R', I')$ (where $R'$ and $I'$ are defined as in Theorem 2), the gcd of $R'(\alpha, x)$ and $I'(\alpha, x)$, denoted $G(\alpha, x)$, has exactly one root (i.e., there exists only one intersection point above each $t = \alpha$).

In order to check the above genericity condition, we have to factorize the roots of the resultant $\text{Res}_x(R', I')$ with respect to the degree of $G(\alpha, x)$. Such an operation is referred to as a *triangular decomposition* of the system $(R', I')$. It yields a set of triangular systems of the form $\{r_k(t), \text{Sres}(t, x)\}_{k=1, \ldots, m}$ such that $\prod_k r_k(t) = \text{Res}_x(R', I')$ and $\text{Sres}(\alpha, x)$ of degree $k$ in $x$ is equal to $G(\alpha, x)$. The polynomial $\text{Sres}_k(t, x) = \sum_{i=0}^k \text{sr}_{k,i} x^i$ is known as the $k$-th subresultant polynomial of $R'$ and $I'$ with respect to $x$ (see [12]). For details on the computation of such a decomposition, we refer to [13].

Once a triangular decomposition computed, the genericity condition is equivalent to the fact that $\text{Sres}_k(t, x)$ can be written modulo $r_k$ as $(a_k(t)x - b_k(t))^k$ with $\gcd(a_k, b_k) = 1$. To check the latter condition, we can use the following result of which the proof can be found in [14].

**Theorem 3:** Let $R(\omega, x)$, $I(\omega, x) \in \mathbb{Q}[\omega, x]$. Define the polynomials $R'(t, x)$, $I'(t, x)$ as in Theorem 2, and let $\{r_k(t), \text{Sres}(t, x)\}_{k=1, \ldots, m}$ be the triangular decomposition of $(R', I')$. Then, $\omega + a x$ is a separating element for $V((R, I))$ if and only if we have

$$k(k - i) \text{sr}_{k,i} \text{sr}_{k,k} - (i + 1) \text{sr}_{k,k-1} \text{sr}_{k,k+1} = 0 \mod r_k,$$

for all $k \in \{1, \ldots, m\}$ and for all $i \in \{0, \ldots, k - 1\}$.

Provided that the condition of Theorem 3 is satisfied (i.e., the system is in generic position), each $\text{Sres}(t, x)$ is then of the form $(a_k(t)x - b_k(t))^k$ with $\gcd(a_k, b_k) = 1$. Moreover, we can obtain the explicit expression of $a_k$ and $b_k$ by computing the $k$-th derivative polynomial of $\text{Sres}(t, x)$, which yields that $a_k = k! \text{sr}_{k,k}$ and $b_k = (k - 1)! \text{sr}_{k,k-1}$. Finally, the solutions of the system $(R, I)$ can be parametrized using the following set of parametrizations:

$$\begin{align*}
    r_k(t) &= 0, \\
    x &= \frac{\text{sr}_{k,k-1}(t)}{k \text{sr}_{k,k}(t)}, \quad k = 1, \ldots, m. \\
    \omega &= t - a \frac{\text{sr}_{k,k-1}(t)}{k \text{sr}_{k,k}(t)}.
\end{align*}$$

**Numerical approximations.** Once a RUR of the solutions of (5) is computed, we can obtain certified numerical approximations of these solutions by first isolating the real roots of the univariate polynomial $f_h$ by means of intervals using, for instance, the algorithm in [10], and then substituting these intervals into the rational fractions $\frac{g_k(t)}{g_0(t)}$ and $\frac{g_0(t)}{g_0(t)}$ in order to get isolating intervals for the coordinates $\omega$ and $x$ of the solutions. Moreover, substituting these intervals into (6) yields intervals for the delays corresponding to each solution.

Efficient algorithms for computing the RUR (8) and the parametrization (9) are implemented in Maple (by means of the command Rational Univariate Representation of the package Groebner for the first one, and an external library RS$^3$). Moreover, the numerical approximations of the solutions, obtained by means of a RUR, can be computed using the Maple command Isolate of the package RootFinding with the option Method="RS" and Output="interval".

**Example 1:** Let us consider the following quasipolynomial used in [1, example 4.6]:

$$f(s, \tau) = e^{3 \tau s - 3} e^{-2 \tau s} + 3 e^{-\tau s} + s^4 + 2 s^2.$$ 

To compute the critical pairs, we consider the following system, which results from the substitutions $s = i \omega$ and $e^{-\tau i} \omega = \frac{s}{s+i}$ into $f(s, \tau)$:

$$\begin{align*}
    R(\omega, x) &= (x^3 - 3 x) \omega^4 + (-2 x^3 + 6 x) \omega^2 + x^3 - 3 x = 0, \\
    I(\omega, x) &= (3 x^2 - 1) \omega^3 + (-6 x^2 + 2) \omega + 3 x^2 + 7 = 0,
\end{align*}$$

and the following polynomial (see Remark 1):

$$f(i \omega, 2 k \pi) = (\omega^2 - 1)^2.$$ 

The latter polynomial yields the critical pairs $(i, 2 k \pi)$ and $(-i, 2 k \pi)$ where $k \in \mathbb{Z}$. Note that the critical imaginary roots have multiplicity two.

Computing a RUR of the former system with respect to the separating linear form $h = \omega + x$, we obtain:

1http://vegas.loria.fr/rs/
\[
\begin{align*}
    f_h &= (t^4 - 2t^2 - 7) (t^8 - 16t^6 + 74t^4 - 32t^2 + 25), \\
    g_h &= t (t^{10} - 15t^6 + 66t^4 - 34t^2 + 143t + 29), \\
    g_{h,\omega} &= t^{10} - 9t^8 - 26t^6 + 234t^4 + 53t^2 + 35, \\
    g_{s,h} &= (t^4 - 2t^2 - 7) (t^6 - 10t^4 + 17t^2 - 10).
\end{align*}
\]

Isolating the roots of the polynomial \( f_h \), we obtain the two real roots
\[
t_1 \in [-67229524645, 34359738368], \quad t_1 \approx -1.956636687, \\
t_2 \in [67229524645, 34359738368], \quad t_2 \approx 1.956636687,
\]
which, after substitution in the rational fractions of the RUR, yields \( x_1 = x_2 = 0 \) and thus \( \omega_i = t_i \) for \( i = 1, 2 \).

Note that the two \( \omega \)-coordinates are roots of the polynomial \( t^4 - 2t^2 - 7 \) which can be solved symbolically. We then get \( \omega = (-1)^i \sqrt{1 + 2\sqrt{2}} \).

Finally, the critical delays \( \tau_{i,k} \) can then be obtained by (6). In our case, the solutions \( (\omega_1, x_1) \) and \( (\omega_2, x_2) \) yields to \( \tau_{i,k} = k \omega_i \), where \( k \in \mathbb{Z} \) and \( i = 1, 2 \).

Alternatively, we can compute the set of parametrizations of the solutions (9). Since all these solutions are simple (i.e., their multiplicities are equal to 1), this yields the following single parametrization:
\[
\begin{align*}
    r_1 &= (t^4 - 2t^2 - 7) (t^8 - 16t^6 + 74t^4 - 32t^2 + 25), \\
    s_{r,1,0} &= 65536 t (3t^6 - 18t^4 + 7t^2 + 18), \\
    s_{r,1,1} &= -327680 t^6 + 1179648 t^4 + 983040 t^2 + 655360.
\end{align*}
\]

The two algorithms, based respectively on the computation of (8) and (9), were implemented in \texttt{Maple}. Below, we report their running times (on a standard laptop with an Intel i7 processor and L8 cache) for randomly generated quasipolynomials of total degree \( \text{deg} \) and density \( \text{dens} \) (sparse/dense).

\begin{center}
\begin{tabular}{cccccc}
\hline
\text{deg} & \text{dens} & \text{numsol} & \text{timing for (8)} & \text{timing for (9)} \\
\hline
5 & 0.5 & 8 & 0.05 & 0.05 \\
5 & 1 & 8 & 0.05 & 0.05 \\
10 & 0.5 & 13 & 1.17 & 0.45 \\
10 & 1 & 15 & 1.62 & 0.6 \\
15 & 0.5 & 28 & 26.58 & 2.8 \\
15 & 1 & 28 & 24.32 & 3.7 \\
20 & 0.5 & 32 & 166.40 & 9.5 \\
20 & 1 & 37 & 182.37 & 14.3 \\
40 & 0.5 & 92 & / & 300 \\
\hline
\end{tabular}
\end{center}

In the table, timings are the running times in CPU seconds and \text{numsol} is the average number of critical pairs \( (\omega, x) \).

**III. COMPUTING PUISEUX SERIES**

This section is organized as follows. We first provide definitions and algorithms for the case of polynomials. This situation is well-studied in the computer algebra literature. See [15], [16], [17] for instance. We then explain how these results can be adapted to the case of quasipolynomials. As shown in [1], [2], these results can be used to effectively decide the stability of LTI differential time-systems with commensurate delays when \( \tau \) changes.

**Definition (polynomial case).** Let us consider a bivariate polynomial \( F \in \mathbb{K}[X, Y] \), where \( \mathbb{K} \) is a field of characteristic 0 (e.g., \( \mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C} \)). Let \( \mathbb{K} \) denote the algebraic closure of \( \mathbb{K} \). If we consider the roots of \( F \) viewed as a univariate polynomial in \( Y \), then we have the following well-known Puiseux theorem.

**Theorem 1:** Let \( F \in \mathbb{K}[X, Y] \) and \( d \) the degree of \( F \) in \( Y \). Given \( x_0 \in \mathbb{K} \), there exist positive integers \( e_1, \ldots, e_s \) with \( \sum_{i=1}^s e_i = d \) and \( d \) distinct series
\[
S_{ij}(X) = \sum_{k=n_i}^{\infty} \alpha_{ik} \zeta_{e_i}^k (X - x_0)^{\frac{k}{e_i}},
\]
where \( 1 \leq i \leq s, 0 \leq j \leq e_i - 1, n_i \in \mathbb{Z}, \zeta_{e_i} \) is the \( e_i \)-th root of unity, and \( \alpha_{in_i} \neq 0 \) if \( S_{ij} \neq 0 \), such that:
\[
F(X, S_{ij}(X)) = 0.
\]

Over most points \( x_0 \), the polynomial \( F(x_0, Y) \) has \( d \) distinct roots. In such a case, the Implicit Function Theorem ensures that the series solutions are actually Taylor series. Such series can be quickly computed using, e.g., quadratic Newton iterations [18]. When dealing with multiple roots over a point \( x_0 \), this may not be the case anymore.

**Computing the first term of the Puiseux series.** We will compute the Puiseux series using a variant of the well-known Newton-Puiseux algorithm [19], [20], namely the rational Newton-Puiseux algorithm [15]. Let us explain the main idea of this algorithm by means of an example, beginning with the computation of the first term of each Puiseux series.

**Example 2:** Let us compute the Puiseux series at \( x_0 = 0 \) of the following polynomial:
\[
F := Y^6 + Y^5 X + 5 Y^4 X^3 - 2 Y^4 X + 4 Y^2 X^2 + X^5 - 3 X^4.
\]

Note that we can always be in this situation via a change of the variable \( X \leftarrow X + x_0 \). From Theorem 1, we know that the first term of any such series \( S(X) \) is of the form:
\[
\alpha X^{\frac{m}{q}}, \quad \alpha \in \mathbb{K}, \quad (m, q) \in \mathbb{N}^2.
\]

We then have:
\[
F(X, \alpha X^{\frac{m}{q}} + \cdots) = \alpha^6 X^{\frac{5m}{q} + 1} + \alpha^5 X^{\frac{4m}{q} + 1 + 1} + 5 \alpha^4 X^{\frac{4m}{q} + 3} - 2 \alpha^4 X^{\frac{2m}{q} + 1} + 4 \alpha^2 X^{\frac{2m}{q} + 2} + X^5 - 3 X^4 + \cdots.
\]

To get \( F(X, Y(X)) = 0 \), at least two terms of the above must cancel, i.e. \( (m, q) \) must be chosen so that two of the above exponents coincide.

This leads us to consider the support of a polynomial \( F \).

**Definition 1:** Given \( F(X, Y) = \sum_{i,j} a_{ij} X^i Y^j \), then
\[
\text{Supp}(F) = \{(i, j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\}
\]
is called the support of \( F \).
The characteristic polynomial of the system. This can actually be done by applying the strategy of the previous paragraph to
\[ F\left(X,Y + \xi \frac{1}{q} X^\frac{1}{q}\right) \in \mathbb{K}\left(\xi^\frac{1}{q}\right)\left[X^\frac{1}{q},Y\right], \]
or similarly to:
\[ F(X^q,Y + \xi \frac{1}{q} X^m) \in \mathbb{K}\left(\xi^\frac{1}{q}\right)[X,Y]. \]
We can also avoid taking a $q$-th root of $x$ by considering the following polynomial [15, section 4]
\[ F(\xi^u X^q, X^m (Y + \xi^v))/X^l \in \mathbb{K}(\xi)[X,Y], \]
where $u,v \in \mathbb{Z}$ are such that $u q - m v = 1$.

### Example 5:

If $f(s,\tau)$ is a quasipolynomial, then we have the following “correspondence”:
\[ \frac{1}{2!} \frac{\partial^2 f}{\partial s^2} (\tau_0, s_0) \Leftrightarrow \text{“coefficient” in } (s - s_0)^2 (\tau - \tau_0). \]
For higher order terms, we can apply a similar idea: instead of evaluating the derivatives at a point, we evaluate
it at the truncated Puiseux series already computed. Details about this strategy will be developed in a forthcoming work.

Finally, we have to change the condition in line 6 in RNPuiseux. The aim of RNPuiseux is to desingularise the curve $F(X, Y) = 0$, i.e. to stop when the multiplicity of the root $\zeta$ is 1. In our context, we might have to stop before such a condition is fulfilled or might have to consider more terms. We stop if and only if the computed coefficient has a non-zero real part.

A symbolic-numeric strategy. In [21], [16], a symbolic-numeric strategy was developed to compute a numerical approximation of the Puiseux series coefficients with correct exponents. Roughly speaking, the idea is to first compute the Puiseux series modulo a well-chosen prime number, which gives the structure (exponents...) of the series, and then to use this structure to conduct purely numeric operations. Adapting this strategy to the quasi-polynomial case is an interesting challenge which will be studied in a future work.

Some examples. We implemented a Maple prototype to compute the Puiseux series of a quasipolynomial as explained above. We illustrate the results with examples and we show the logs of some computations (including the timing).

Example 6: We consider again the quasipolynomial defined in Example 1 and compute its Puiseux series at $(\tau_0, s_0) = (2\pi, i)$. We get the following:

Starting; evaluation point is 2*Pi I
Newton polygon is [[0, 2], [1, 2], [0, 0]]
xi is 1/4*I
Real part of the coefficient is 1/4*z2*(1/2)
We’re done; it took .88e-1 seconds

Hence, we obtain:

$$\Delta s = (0.3535 + 0.3535 i) \Delta \tau_2.$$ 

Example 7: We consider again the quasipolynomial from Example 6 and compute its Puiseux series at $(\tau_0, s_0) = (2\pi, i)$. We get the following:

Starting; evaluation point is 2*Pi I
Newton polygon is [[0, 3], [2, 0]]
xi is -1/4*I
Real part of the coefficient is 1/4*z2*(1/2)
We’re done; it took .88e-1 seconds

Hence, we obtain:

$$\Delta s = \sum_{k=0}^{\infty} a_k(s) e^{-k s \tau}$$ with $a_k(s) = s^4 + 2 s^2 + 2 s + 1, a_3(s) = 3 s^4 + 9 s^3 + 9 s + 4, a_2(s) = 5/4 s^4 + 11/4 s^3 + 3 s^2 + 1/2 s^2 \pi + 13 s^2 + 15 s - 9/4 s \pi + 6 - 9/4 \pi, a_1(s) = 5/4 s^3 + 11/2 s^2 + s^3 + 7/2 s^2 + 3 s^2 \pi + 7 s^2 + 11 s + 9/4 s \pi + 4 - 9/2 \pi$ and $a_0(s) = 1 + 3 s + 9/2 s^3 + s^2 + 45/8 s^2 + 2/9 s \pi - 15/8 s^2 \pi^2 + 1/2 s^2 \pi - 75/8 s^2 \pi^2 - 9/4 s + 11/4 s^2 + 15/8 s^2 \pi^2$.

Computing the Puiseux series at the point $(5 \pi, i)$, we get:

Newton polygon is [[0, 2], [1, 2], [0, 0]]
Edge 1, xi is -((1/2) *I)/Pi
Real part is 0; Going for a recursive call
Newton polygon is [[0, 1], [1, 0]]
xi is (-3/208+1/104)*I*(117*Pi+2-3*I)/Pi^2
Real part is not zero
We’re done
Edge 2, xi is ((6/5)*I)/(50*Pi^3-30*Pi^2-3*Pi)
Real part is 0; Going for a recursive call
Newton polygon is [[0, 1], [1, 0]]
xi is (see below)
real part is non zero
We’re done
We’re done; it took 3.596 seconds

This leads to the following Puiseux series:

$$\Delta s = -0.1591 i \Delta \tau - (0.5371 - 0.3644 i) \Delta \tau^2$$

$$\Delta s = -0.0987 i \Delta \tau^{1/3} - (0.03557 - 0.0028 i) \Delta \tau^{2/3}$$

This example is the last example, the coefficient growth is important; for instance, the root $\zeta$ that is not written above is actually

$$3 \pi = 200 i \pi^2 - 180 i \pi^3 - 750 \pi^2 - 21 i + 600 \pi^2$$

This explains why the running time is longer on this example. This advocates for the development of an efficient symbolic-numeric strategy.

IV. Appendix

A key concept for the study of the solutions of polynomial systems is the so-called subresultant sequence. Below, we give its definition and some of its properties that are used in our algorithms. For a complete overview, we refer to [12].

Let $A$ be a unique factorization domain (e.g., $A := \mathbb{K}[y]$) and $F$ its fraction field, i.e., $F := \{ g/y | 0 \neq d, n \in A \}$. Let $f = a_0 + \ldots + a_n x^n$ and $g = b_0 + \ldots + b_m x^m$. When computing the Puiseux series, we have a subresultant sequence $\tau, \pi$. In the following, we assume that $a_n \neq 0, b_n \neq 0$ and $n \geq m$.

For an integer $k$ such that $0 \leq k \leq m$, we can define the following $A$-linear map

$$\varphi_k : (u, v) \in A^{k+2} \rightarrow u f + v g \in A,$$

where such that $u, v \in A[x]$ are polynomials of degree respectively strictly less than $m - k - 1$ and $n - k - 1$.

Using the canonical basis $\{ x^i \}_{i=0,\ldots,k}$ of the free $A$-module formed by the polynomials of degree less than or equal to $k$,
\( \varphi_k \) can be represented by means of the the following matrix

\[
S_k := \begin{pmatrix}
  a_n & a_{n-1} & \cdots & a_0 \\
  \vdots & \vdots \\
  a_n & a_{n-1} & \cdots & a_0 \\
  b_m & b_{m-1} & \cdots & b_0 \\
  \vdots & \vdots \\
  b_m & b_{m-1} & \cdots & b_0
\end{pmatrix}
\]

formed by \( n + m - 2 \times k \) rows and \( n + m - k \) columns.

The matrix \( S_0 \) is the standard Sylvester matrix associated with \( f \) and \( g \). To be coherent with the degree of polynomials, we will attach index \( i - 1 \) to the \( i \)-th column of \( S_k \) so that the index of the columns goes from \( 0 \) to \( n + m - k - 1 \).

**Definition 4:** For \( j \leq n + m - k - 1 \) and \( 0 \leq k \leq m \), let \( sr_{j,k} \) be the determinant of the submatrix of \( S_k \) formed by the last \( n + m - 2 \times k - 1 \) columns, the column of index \( j \) and all the \( n + m - 2 \times k \) rows. The following polynomial

\[
Sres_k(f, g) = sr_{k,k}x^k + \ldots + sr_{k,0}
\]

is the \( k \)-th polynomial subresultant of \( f \) and \( g \), and its leading term \( sr_{k,k} \) is the \( k \)-th principal subresultant of \( f \) and \( g \). The polynomial \( \text{Res}(f,g) = sr_{0,0} \) is the resultant of \( f \) and \( g \).

**Theorem 4:** The following assertions are equivalent:

- \( f \) and \( g \) have a common root in \( \mathbb{F} \).
- \( \text{Res}_0(f, g) = 0 \).
- \( f \) and \( g \) have a non trivial gcd which is proportional to the non-zero polynomial subresultant of minimal index.

**Theorem 5:** Let \( \mathcal{A} \) and \( \mathcal{A}' \) be unique factorization domains and \( \phi : \mathcal{A} \to \mathcal{A}' \) be a morphism. Let \( f, g \in \mathcal{A}[x] \) such that \( \deg(\phi(f)) = \deg(f) \geq \deg(g) = \deg(\phi(g)) \). Then:

\[
\phi(Sres_i(f, g)) = Sres_i(\phi(f), \phi(g)), \quad i = 0, \ldots, \deg(g).
\]

Consider now two polynomials \( f = \sum_{i=0}^{m} a_i(x_1) x_2^i \) and \( g = \sum_{i=0}^{m} b_i(x_1) x_2^i \) in \( \mathbb{Q}[x_1, x_2] \). The two above properties leads to the following result.

**Theorem 6:** For any \( \alpha \) such that \( a_n(\alpha) \) and \( b_m(\alpha) \) do not both vanish. The first polynomial \( Sres_{x_2, k}(\alpha, x_2) \) (for \( k \) increasing) that does not identically vanish is of degree \( k \) and is the gcd of \( f(\alpha, x_1) \) and \( g(\alpha, x_2) \) (up to a nonzero constant in the fraction field of \( \mathbb{K}(\alpha) \)).

**Triangular description of the solutions.**

Given two bivariate polynomials \( f, g \in \mathbb{Q}[x_1, x_2] \) such that \( a_n(x_1) \) and \( b_m(x_1) \) are coprime, one can use the above result in order to compute a triangular description of the solutions of the system \( S := \{ f = g = 0 \} \). Indeed, starting from the resultant of \( f \) and \( g \) whose roots \( \alpha \) are the \( x_1 \)-coordinates of the common solutions of \( S \), one can factorize the latter depending on the degree of the gcd of \( f(\alpha, x_2) \) and \( g(\alpha, x_2) \). For each root \( \alpha \) of the resultant, the gcd of \( f(\alpha, x_2) \) and \( g(\alpha, x_2) \) is then given as the specialization at \( \alpha \) of the first non-vanishing polynomial subresultant according to Theorem 4. Consequently, the set of solutions of \( S \), i.e., \( \{ (\alpha, \beta) \in \mathbb{C}^2 | f(\alpha, \beta) = g(\alpha, \beta) = 0 \} \) is equal to the set \( \bigcup_{i=1}^{\infty} \{ (\alpha, \beta) \in \mathbb{C}^2 | r_i(\alpha) = Sres_0(\alpha, \beta) = 0 \} \), where the polynomial \( r_i(x_1) \) is the factor of the resultant whose roots \( \alpha \) satisfy the property that the degree of the gcd of \( f(\alpha, x_2) \) and \( g(\alpha, x_2) \) is equals to \( i \). For more details about this triangular decomposition algorithm, see [13].