Connectivity Queries on Curves in $\mathbb{R}^n$

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Abstract

We study connectivity questions for algebraic curves in n-dimensional space. A suitable description of an algebraic curve $\mathscr{C}$ in $\mathbb{C}^n$ and the coordinates of two points A, B on $\mathscr{C} \cap \mathbb{R}^n$ are given as input, we wish to decide whether A and B are on the same connected component of $\mathscr{C} \cap \mathbb{R}^n$.

$\mathscr{C}$ will be given by means of a *one-dimensional parametrization*.

1 Motivation

Roadmaps(Canny's notion) allow one to reduce connectivity queries on semi-algebraic sets to connectivity queries on curves.

Let n be the dimension of the ambient space, and let $X_1, \dots, X_n$ be coordinates in $\mathbb{C}^n$. Following Canny's algorithm, and improvements by Basu, Pollack and Roy [1], the roadmap algorithm from [5] computes the following:

1. two linear forms $X_1 = \eta_1 X_1 + \cdots + \eta_n X_n$ and $X_2 = \vartheta_1 X_1 + \vartheta_2 X_2 + \cdots + \vartheta_n X_n$, with coefficients in $\mathbb{Q}$
2. polynomials $q, q_0, \dots, q_n$ in $\mathbb{Q}[X_1, X_2]$ where X_1 and X_2 are indeterminates.

Let $Z \subset \mathbb{C}^n$ be the constructible set defined by

$$q(X_1, X_2) = 0, \qquad X_i = \frac{q_i(X_1, X_2)}{q_0(X_1, X_2)} \quad (1 \le i \le n), \qquad q_0(X_1, X_2) \ne 0.$$

Then, the roadmap R is obtained as $C \cap \mathbb{R}^n$, where $C \subset \mathbb{C}^n$ is the algebraic curve obtained as the Zariski closure of Z.

- Roadmap i.e. $(q, q_0, \dots, q_n)$ and X_1, X_2, two query points A, B on R are input
- Question is to decide whether A and B are on the same connected component of R.
- Close reference is in [6], uses regular chain as more general input and relies on Puiseux series computations.

2 Preliminaries

One-dimensional Parametrization A family of polynomials $\mathscr{Q} = (Q, Q_0, Q_3, \dots, Q_n)$ forms a *one-dimensional parametrization defined over* $\mathbb{K}$ if they are in $\mathbb{K}(X_1)[X_2]$, with Q squarefree, monic of positive degree in X_2, with $\gcd(Q, Q_0) = 1$ and with $\deg(Q_i, X_2) < \deg(Q, X_2)$ for all i. We associate to $\mathscr{Q}$ the set $Z(\mathscr{Q}) \subset \overline{\mathbb{K}}^n$ defined by

$$\Delta(X_1) Q_0^\star(X_1, X_2) \ne 0, \quad Q^\star(X_1, X_2) = 0, \quad X_i = \frac{Q_i^\star(X_1, X_2)}{Q_0^\star(X_1, X_2)} \quad (3 \le i \le n), \tag{1}$$

and we let $\mathfrak{Z}(\mathscr{Q}) \subset \overline{\mathbb{K}}^n$ be the Zariski closure of $Z(\mathscr{Q})$.

Curves in n-dimensional space. Let $\mathscr{C}$ be an algebraic curve in $\overline{\mathbb{K}}^n$ such that the following holds:

(H_1) the projection of each irreducible component of $\mathscr{C}$ on the X_1-axis is dense in $\overline{\mathbb{K}}$;

(H_2) $\mathscr{C}$ is birationnally equivalent to its projection on the (X_1, X_2)-plane.

This is easily seen to be equivalent to the existence of a one-dimensional parametrization $\mathscr{Q}$ such that $\mathfrak{Z}(\mathscr{Q}) = \mathscr{C}$.

3 Genericity Requirements

Algorithm needs input curve to be in generic position. Thus following assumptions(as a whole called H) are to be satisfied:

(H$_1$) The restriction of Π_i to $\mathscr{C}$ is proper for $1 \le i \le n$.

(H$_2$) The restriction of Π_2 to $\mathscr{C}$ is birational.

(H$_3$) The set $\mathrm{crit}(\Pi_1, \mathscr{C})$ has dimension at most 0 and for all $\mathbf{x} \in \mathrm{crit}(\Pi_1, \mathscr{C}) \cap \mathrm{reg}(\mathscr{C})$, $\dim(\Pi_2(T_{\mathbf{x}}\mathscr{C})) = 1$.

(H$_4$) The restriction of Π_3 to $\mathscr{C}$ is an isomorphism between $\mathscr{C}$ and $\Pi_3(\mathscr{C})$.

(H$_5$) If $\mathbf{x} \in \mathscr{C}$ is a plane singularity, then its tangent plane does not contain the line defined by $X_1 = \alpha, X_2 = \beta$ where $(\alpha, \beta) = \Pi_2(\mathbf{x})$

If the curve or its representation satisfy the above assumptions then it satisfies generic requirement.

Theorem 1. *There exists a proper Zariski-closed set* $\mathscr{F} \subset \mathrm{GL}_n(\overline{\mathbb{K}})$ *such that for all* $\mathbf{A} \in \mathrm{GL}_n(\mathbb{K}) \setminus \mathscr{F}$, $\mathscr{C}^{\mathbf{A}}$ *satisfies assumption* H.

4 Change of Coordinates

We have to change coordinates to put our curve in generic position. Algorithm for change of coordinates is probabilistic. Change of variable is used as input, is also probabilistic. We do it as follows:

4.1 Change of variable over $\mathbb{K}$

Change of $X_3 \cdots X_n$ We use linear combination of $(Q_3 \cdots Q_n)$ to change $X_3 \cdots X_n$.

Change of X_2 We use evaluation and interpolation. For any specialization value x_0 of X_i the time required is given by following lemma:

Lemma 2. *Suppose that the characteristic of* $\mathbb{K}$ *is greater than* δ, *where* $\delta = \deg(\mathscr{Q})$. *Then it requires* $O(n\mathsf{C}(\delta))$ *in time.*

We define $\mathsf{C}(\mathbf{n})$ as no. of operations needed to compute $K = G(H) \bmod F$, given F, G, H in $\mathbb{K}[X]$, with $\deg(F) = n$ and $\deg(G), \deg(H) < n$. And $\mathscr{Q} = (Q, Q_2, \dots, Q_n)$ is normalized parametrization of the curve in generic position.

4.2 Change of variable over $\mathbb{K}(X_1)$

We do the same for changing X_1 as we do for X_2. The overall running time is given by the following proposition.

Proposition 3. *Let* $\varepsilon \le 1$ *be a positive real number. If the characteristic of* $\mathbb{K}$ *is at least equal to* $L = \lceil 9\Delta^3/\varepsilon \rceil$, *then with probability at least* $1 - \varepsilon$, *it requires* $O(n\delta\mathsf{C}(\delta))$;

4.3 Change of variables over $\mathbb{Q}(Y)$

Considering base field $\mathbb{Q}(Y)$ our main concern is to determine bit complexity. We follow the same approach with chinese remaindering technique. The tricky part is the analysis of bit complexity and its still a work in progress.

5 Our algorithm

The algorithm we propose is inspired by El Kahoui's algorithm [4] and also use ideas of isolating boxes from Seidel and Wolport [7] and Cheng et. el. [3]. Change of variable over $\mathbb{K}$ helps us to make changes in coordinates for the input curve and place it in generic position. Supposing that the chosen change of coordinates is generic, our algorithm works in three steps:

1. Compute a rational parametrization $(q', q'_0, \dots, q'_n), X_1, X_2$ of $C^{\mathbf{A}} = \{\mathbf{A}z \mid z \in C\}$ using fast algorithms for change of orderings in triangular sets;
2. Computes the topology of the plane curve defined by $q'(X_1, X_2) = 0$ using e.g. [3];
3. Consider the space curve defined as the Zariski-closure of the constrictible set defined by

$$q'(X_1, X_2) = 0, \quad X_3 = \frac{q'_3(X_1, X_2)}{q'_0(X_1, X_2)}, \quad q_0(X_1, X_2) \ne 0.$$

Using results from [4], deduce the topology of the space curve from the one computed in Step 1, and use it to answer connectivity queries on the space curve.

6 Benchmarks

We implemented our algorithm stepwise in maple. Step 1 gives us a rational bivariate representation(RBR) of parametrization. We generated a few examples to test. Step 2 consists of several tasks like computing Isolated boxes, creating vertices, connectivity etc. We are done with computing Isolated boxes and currently working on connectivity We test our RBRs with ComputeIsolateBoxes program and took timing. In the table given below

- rbr-i-sep means initial system has i no. of variables and parametrizations(describing the curve) are seperate, independant of each other.
- rbr-i-uni means initial system has i no. of variables and only one parametrization describing the curve.
- t_{rbr} time required to compute RBR
- t_{iso} time to compute Isolated boxes from RBR
- x sec: work in progress

RBR	degree	no. vars	t_{rbr}	t_{iso}
rbr-3-sep	6	3	.331	3.71 sec
rbr-3-uni	6	3	.355	x sec
rbr-4-sep	18	4	4.679	x sec
rbr-4-uni	18	4	4.963	x sec
rbr-5-sep	42	5	216.790	x sec

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