Theoretical Computer Science 178 (1997) 275–283

Theoretical Computer Science

Note

Recognizable picture languages and domino tiling[1]

Michel Latteux, David Simplot*

C.N.R.S. U.A. 369, L.I.F.L., Université de Lille 1, 59655 Villeneuve d'Ascq, Cedex, France

Communicated March 1996; revised September 1996
Communicated by M. Nivat

Abstract

In [2], Giammarresi and Restivo define the notion of local picture languages by giving a set of authorized 2×2 tiles over $\Sigma \cup \{\#\}$ where # is a boundary symbol which surrounds the pictures. Then they define the class of recognizable picture languages as the set of languages which can be obtained by projection of a local one. This class is of interest since it admits several quite different characterizations [3]. Here, we define the hv-local picture languages where 2×2 tiles are replaced by horizontal and vertical dominoes. So the horizontal and the vertical scanning can be done separately. However, we prove that every recognizable picture language can be obtained as a projection of a hv-local language.

1. Introduction

Local sets of words play a considerable role in the theory of recognizable string languages. For example, it is well known that every recognizable subset of Σ^+ can be obtained as the image of a local set by a letter-to-letter morphism [1]. A local set L over the alphabet Δ can be defined by a subset A of $(\Delta \cup \{\#\})^2$ which indicates the authorized consecutions of letters in a word of L: ω is in L iff every factor of length 2 in $\#\omega\#$ belongs to A. These factors can be seen as dominoes that permit to scan $\#\omega\#$.

In [2, 3] Gimmarresi and Restivo generalize this notion to picture languages that are sets of rectangular arrays of symbols. In order to identify the symbols that are on the border, each picture is surrounded with the boundary symbol # and the local scanning is defined in terms of a finite set of square pictures of dimension 2. This yields, via a projection, to a very simple definition of the family of recognizable picture languages

* Corresponding author. E-mail: simplot@lifl.fr.

[1] This work was partially supported by "PRC/GDR Mathématiques et Informatique" and ESPRIT Basic Research Action 6317 ASMICS 2.

PII S0304-3975(96)00283-6

which reveals of great interest since it corresponds to a family defined in terms of a particular kind of cellular automata [5, 6] and coincides with the family of existential monadic second-order definable picture languages [7, 3]. With the 2×2 tiles used in the definition of local picture languages, the horizontal and vertical scanning are mixed. It is, then, natural to wonder whether these two scannings can be done separately. So, we define hv-local picture languages by replacing in the definition of local picture languages the 2×2 tiles by horizontal and vertical dominoes and we prove that every recognizable picture language can be obtained as the projection of some hv-local picture language.

2. Recognizable picture languages

Let Σ be a finite alphabet. A picture over Σ is a two-dimensional rectangular array of letters of Σ. The set of all the pictures over Σ is denoted Σ^{**}.

If p is a picture, $row(p)$ and $col(p)$, respectively, denote the number of rows and number of columns of p. Note that we consider only non-empty arrays: the number of columns and of rows are always greater than 0. The size of p, denoted $size(p)$, is the couple $(row(p), col(p))$. For every $1 \leqslant i \leqslant row(p)$ and $1 \leqslant j \leqslant col(p)$, $p(i,j)$ denotes the letter (of Σ) which is in ith row and jth column (starting on the left-bottom corner). The set of all the pictures over Σ of size (m,n) is denoted $\Sigma^{m,n}$.

If p is a picture over Σ of size (m,n), we note $\tilde{p}$, the $(m+2, n+2)$ picture over $\Sigma \cup \{\#\}$ with # a special letter which do not belong to Σ, and such that

(i) $\forall 1 \leqslant i \leqslant m+2,\ \tilde{p}(i,1) = \tilde{p}(i,n+2) = \#$,

(ii) $\forall 1 \leqslant j \leqslant n+2,\ \tilde{p}(1,j) = \tilde{p}(m+2,j) = \#$,

(iii) $\forall 2 \leqslant i \leqslant m+1,\ 2 \leqslant j \leqslant n+1,\ \tilde{p}(i,j) = p(i-1, j-1)$.

For example, if we consider the picture p of size $(3,4)$ over $\Sigma = \{0,1,2\}$:

$p =$

1	0	2	0
0	1	0	0
0	0	2	1

we get

$\tilde{p} =$

#	#	#	#	#	#
#	1	0	2	0	#
#	0	1	0	0	#
#	0	0	2	1	#
#	#	#	#	#	#

If p is a picture over Σ of size (m,n), $r \leqslant m$ and $s \leqslant n$, $T_{r,s}(p)$ is the set of the (r,s) subpictures of p. We define

$$T_{r,s}(p) = \{q \in \Sigma^{r,s} \mid \exists 0 \leqslant x \leqslant m-r,\ 0 \leqslant y \leqslant n-s\ \forall 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\ \ q(i,j) = p(x+i, y+j)\}.$$

A picture language over Σ is a subset of Σ^{**}. Let L be a picture language. We define $T_{r,s}(L) = \bigcup_{p \in L} T_{r,s}(p)$.

Definition 1. Let L be a picture language over Σ. L is local if there exists a set Δ of $(2,2)$ pictures over $\Sigma \cup \{\#\}$ such that $L = \{p \in \Sigma^{**} \mid T_{2,2}(\tilde{p}) \subseteq \Delta\}$.

For instance, if we consider the picture language L over $\Sigma = \{0,1,2\}$ of all the horizontal lines of thickness 2 obtained by concatenation of the square:

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 2 & 0 \\ \hline \end{array}$$

$$L = \left\{ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 2 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 0 \\ \hline 2 & 0 & 2 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 2 & 0 & 2 & 0 & 2 & 0 \\ \hline \end{array}, \ldots \right\}$$

The picture language L is local, and we can associate it with the set of squares Δ:

$$\Delta = \left\{ \begin{array}{cccc} \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 1 & 0 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & \# \\ \hline \end{array} \\ \begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & 2 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 2 & 0 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 2 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 0 & \# \\ \hline 0 & \# \\ \hline \end{array} \\ \begin{array}{|c|c|}\hline \# & 2 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 2 & 0 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 0 & 2 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 0 & \# \\ \hline \# & \# \\ \hline \end{array} \end{array} \right\}$$

Let Σ and Σ' be two finite alphabets and $\pi : \Sigma \to \Sigma'$ a mapping. The projection by π of a picture $p \in \Sigma^{**}$ is the picture $p' \in \Sigma'^{**}$ such that $size(p) = size(p')$ and for all $1 \leqslant i \leqslant row(p), 1 \leqslant j \leqslant col(p)$ $p'(i,j) = \pi(p(i,j))$. We note $p' = \pi(p)$. By extension, we note $\pi(L)$, the projection by mapping by π of the language L over Σ and $\pi(L) = \{p' \in \Sigma'^{**} \mid \exists p \in L, p' = \pi(p)\}$.

Definition 2. Let L be a picture language over Σ. L is recognizable if there exists a local picture language L' over Σ' and a mapping $\pi : \Sigma' \to \Sigma$ such that $L = \pi(L')$.

For example, if we look at the picture language K over $\Gamma = \{a\}$ of all the lines of thickness 2 and of even width:

$$K = \left\{ \begin{array}{|c|c|}\hline a & a \\ \hline a & a \\ \hline \end{array}, \begin{array}{|c|c|c|c|}\hline a & a & a & a \\ \hline a & a & a & a \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|}\hline a & a & a & a & a & a \\ \hline a & a & a & a & a & a \\ \hline \end{array}, \ldots \right\}$$

K is a recognizable picture language for the mapping $\pi : \Sigma = \{0,1,2\} \to \Gamma$ such that $\pi(0) = \pi(1) = \pi(2) = a$. It is easy to see that $K = \pi(L)$, where L is the local picture language of the previous example.

3. "hv-local" picture languages

Now, we introduce the "hv-local" picture languages where the horizontal and vertical controls are dissociated by using dominoes instead of 2×2 squares.

Definition 3. Let L be a picture language included in Σ^{**}. L is hv-local if there exists a set Δ of horizontal and vertical dominoes over $\Sigma \cup \{\#\}$ such that $L = \{q \in \Sigma^{**} \mid T_{1,2}(\tilde{q}) \cup T_{2,1}(\tilde{q}) \subseteq \Delta\}$.

For instance, if we consider the picture language L over $\Sigma = \{0,1,2\}$ of all the horizontal lines of thickness 1 obtained by concatenation of the domino:

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline\end{array}$$

$$L = \left\{ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 0 \\ \hline\end{array},\ \begin{array}{|c|c|c|c|c|c|}\hline 1 & 0 & 1 & 0 & 1 & 0 \\ \hline\end{array}, \ldots \right\}$$

L is a hv-local picture language, because we can associate the set of dominoes Δ:

$$\Delta = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline \# & \# \\ \hline\end{array},\ \begin{array}{|c|c|}\hline \# & 1 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 0 & \# \\ \hline\end{array} \\ \begin{array}{|c|}\hline \# \\ \hline \# \\ \hline\end{array},\ \begin{array}{|c|}\hline \# \\ \hline 1 \\ \hline\end{array},\ \begin{array}{|c|}\hline 1 \\ \hline \# \\ \hline\end{array},\ \begin{array}{|c|}\hline \# \\ \hline 0 \\ \hline\end{array},\ \begin{array}{|c|}\hline 0 \\ \hline \# \\ \hline\end{array} \end{array} \right\}$$

Proposition 4. *Let $L \subseteq \Sigma^{**}$ be a picture language. If L is hv-local, then L is local.*

Proof. Let $L \subseteq \Sigma^{**}$ be a hv-local picture language. We construct a local picture language K and we show that $L = K$.

We know that there exists a set Δ of horizontal and vertical dominoes over $\Sigma \cup \{\#\}$ ($\Delta \subseteq (\Sigma \cup \{\#\})^{1,2} \cup (\Sigma \cup \{\#\})^{2,1}$) such that $L = \{p \in \Sigma^{**} \mid T_{1,2}(\tilde{p}) \cup T_{2,1}(\tilde{p}) \subseteq \Delta\}$.
We define a set of squares Δ':

$$\Delta' = \{q \in (\Sigma \cup \{\#\})^{2,2} \mid T_{1,2}(q) \cup T_{2,1}(q) \subseteq \Delta\}.$$

Let $K = \{p \in \Sigma^{**} \mid T_{2,2}(\tilde{p}) \subseteq \Delta'\}$. Obviously, K is a local language and we show that $L = K$.

Let $p \in K$. Then $T_{2,2}(\tilde{p}) \subseteq \Delta'$ and $T_{1,2}(\tilde{p}) \subseteq T_{1,2}(T_{2,2}(\tilde{p})) \subseteq T_{1,2}(\Delta') \subseteq \Delta$. Similarly $T_{2,1}(\tilde{p}) \subseteq \Delta$. Hence $p \in L$. On the other hand, let $q \in L$ and $a \in T_{2,2}(\tilde{q})$. Then $T_{1,2}(a) \subseteq T_{1,2}(\tilde{q}) \subseteq \Delta$ and $T_{2,1}(a) \subseteq T_{2,1}(\tilde{q}) \subseteq \Delta$. So $a \in \Delta'$ and $q \in K$. $\square$

For instance, the picture language of the previous example is local with:

$$\Delta' = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & 1 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline \# & \# \\ \hline 1 & 0 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & 1 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & \# \\ \hline\end{array} \\ \begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & \# \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline \# & \# \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 0 & 1 \\ \hline \# & \# \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 0 & \# \\ \hline \# & \# \\ \hline\end{array} \end{array} \right\}$$

Proposition 5. *Let $L \subseteq \Sigma^{**}$ be a picture language. If L is local, there exists a hv-local picture language L' over Σ' and a mapping $\pi : \Sigma' \to \Sigma$ such that $L = \pi(L')$.*

Proof. First, we define an extended alphabet from Σ. We denote this alphabet $ext(\Sigma) = (\Sigma \cup \{\#\})^{3,3}$. We now define a mapping θ from Σ^{**} to $ext(\Sigma)^{**}$:

$$\theta : \Sigma^{**} \to ext(\Sigma)^{**}$$
$$p \mapsto p' \in ext(\Sigma)^{**} \quad \text{with } size(p') = size(p)$$

and

$$\forall 1 \leqslant i \leqslant col(p) \quad \forall 1 \leqslant j \leqslant row(p)$$

$$p'(i,j) = \begin{array}{|c|c|c|} \hline \tilde{p}(i+2,j) & \tilde{p}(i+2,j+1) & \tilde{p}(i+2,j+2) \\ \hline \tilde{p}(i+1,j) & \tilde{p}(i+1,j+1) & \tilde{p}(i+1,j+2) \\ \hline \tilde{p}(i,j) & \tilde{p}(i,j+1) & \tilde{p}(i,j+2) \\ \hline \end{array} \qquad (1)$$

We can note that every 2×2 tiles appearing in $\tilde{p}$, where $p \in \Sigma^{**}$, appear in the letters of $\theta(p)$, and at reverse, every tiles appearing in the letters of $\theta(p)$ are in $\tilde{p}$, i.e.:

$$T_{2,2}(\tilde{p}) = \bigcup_{a \in T_{1,1}(\theta(p))} T_{2,2}(a)\,. \qquad (2)$$

For instance, if we consider the picture $p \in \{0,1\}^{**}$:

$$p = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \quad \text{then, we have :} \quad \theta(p) = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

with

$$a = \begin{array}{|c|c|c|} \hline \# & \# & \# \\ \hline \# & 1 & 0 \\ \hline \# & 0 & 1 \\ \hline \end{array}, \qquad b = \begin{array}{|c|c|c|} \hline \# & \# & \# \\ \hline 1 & 0 & \# \\ \hline 0 & 1 & \# \\ \hline \end{array},$$

$$c = \begin{array}{|c|c|c|} \hline \# & 1 & 0 \\ \hline \# & 0 & 1 \\ \hline \# & \# & \# \\ \hline \end{array}, \qquad d = \begin{array}{|c|c|c|} \hline 1 & 0 & \# \\ \hline 0 & 1 & \# \\ \hline \# & \# & \# \\ \hline \end{array}.$$

We also define a mapping π from $ext(\Sigma)^{**}$ into Σ^{**} by $\pi(a) = a(2,2)$ for $a \in ext(\Sigma)$. By (1), it is obvious that for all $p \in \Sigma^{**}$ we have $p = \pi(\theta(p))$.

Let us consider the picture language $K = \theta(\Sigma^{**})$. We are going to show the following claim.

Claim 6. *The picture language K is hv-local.*

In order to show this claim, we consider the set of dominoes Δ' defined by

$$\Delta' = \{T_{2,1}(\tilde{q}) \mid q \in K\} \cup \{T_{1,2}(\tilde{q}) \mid q \in K\}.$$

It is easy to see that

$$\Delta' = \left\{ \begin{array}{|c|}\hline \# \\ \hline \# \\ \hline \end{array} \right\} \cup V_1 \cup V_2 \cup V_3 \cup \left\{ \begin{array}{|c|c|}\hline \# & \# \\ \hline \end{array} \right\} \cup H_1 \cup H_2 \cup H_3$$

with

$$\begin{cases} V_1 = \left\{ \begin{array}{|c|}\hline \# \\ \hline a \\ \hline \end{array} \,\middle|\, a \in \Sigma' \wedge a(3,1) = a(3,2) = a(3,3) = \# \right\} \\[2ex] V_2 = \left\{ \begin{array}{|c|}\hline a \\ \hline \# \\ \hline \end{array} \,\middle|\, a \in \Sigma' \wedge a(1,1) = a(1,2) = a(1,3) = \# \right\} \\[2ex] V_3 = \left\{ \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \,\middle|\, a, b \in \Sigma' \wedge \forall 1 \leqslant i \leqslant 3 \;\; (a(2,i) = b(3,i) \wedge a(1,i) = b(2,i)) \right\} \end{cases}$$

and

$$\begin{cases} H_1 = \left\{ \begin{array}{|c|c|}\hline \# & a \\ \hline \end{array} \,\middle|\, a \in \Sigma' \wedge a(1,1) = a(2,1) = a(3,1) = \# \right\} \\[1ex] H_2 = \left\{ \begin{array}{|c|c|}\hline a & \# \\ \hline \end{array} \,\middle|\, a \in \Sigma' \wedge a(1,3) = a(2,3) = a(3,3) = \# \right\} \\[1ex] H_3 = \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \,\middle|\, a, b \in \Sigma' \wedge \forall 1 \leqslant i \leqslant 3 \;\; (a(i,2) = b(i,1) \wedge a(i,3) = b(i,2)) \right\} \end{cases}$$

Now, let K' be the hv-local picture language over $ext(\Sigma)$ defined by Δ'. By definition of Δ' and K', the language K is included in K'. In order to prove the reverse inclusion, let us consider a picture $p' \in K'$ and the picture $p = \pi(p')$. We will show that $p' = \theta(p)$, i.e. p' satisfies (1). This proof consists of three steps.

First, since $p = \pi(p')$, we get

$$\forall i,j \quad p'(i,j)(2,2) = \tilde{p}(i+1, j+1). \tag{P1}$$

Secondly, let us consider $p'(i,j)(3,2)$, we have to distinguish two cases:

(i) If $i = row(p')$, the following domino belongs to $T_{2,1}(\tilde{p'}) \subseteq \Delta'$:

$$\begin{array}{|c|}\hline \# \\ \hline p'(i,j) \\ \hline \end{array} \in V_1$$

Therefore we have $p'(i,j)(3,2) = \# = \tilde{p}(i+2, j+1)$.

(ii) If $i < row(p)$, the following domino belongs to $T_{2,1}(\tilde{p'}) \subseteq \Delta'$:

$$\begin{array}{|c|}\hline p'(i+1,j) \\ \hline p'(i,j) \\ \hline \end{array} \in V_3$$

We have $p'(i,j)(3,2) = p'(i+1,j)(2,2)$ and by (P1), we have $p'(i+1,j)(2,2) = \tilde{p}(i+2, j+1)$.

In the same way, for $p'(i,j)(1,2)$, $p'(i,j)(2,1)$ and $p'(i,j)(2,3)$, we show that

$$\begin{aligned} &\forall i,j \;\; p'(i,j)(3,2) = \tilde{p}(i+2,j+1),\\ &\forall i,j \;\; p'(i,j)(1,2) = \tilde{p}(i,j+1),\\ &\forall i,j \;\; p'(i,j)(2,1) = \tilde{p}(i+1,j),\\ &\forall i,j \;\; p'(i,j)(2,3) = \tilde{p}(i+1,j+2). \end{aligned} \tag{P2}$$

At last, let us consider $p'(i,j)(3,1)$. Two different cases can happen:

(i) If $j=1$, the following domino belongs to $T_{1,2}(\tilde{p'}) \subseteq \Delta'$:

$$\boxed{\# \mid p'(i,j)} \in H_1$$

and we have $p'(i,j)(3,1) = \# = \tilde{p}(i+2,j)$.

(ii) If $j>1$, the following domino belongs to $T_{1,2}(\tilde{p'}) \subseteq \Delta'$:

$$\boxed{p'(i,j-1) \mid p'(i,j)} \in H_3$$

and we have $p'(i,j)(3,1) = p'(i,j-1)(3,2)$ and by (P2), we have $p'(i,j-1)(3,2) = \tilde{p}(i+2,j)$.

In the same way, we prove that

$$\begin{aligned} &\forall i,j \;\; p'(i,j)(3,1) = \tilde{p}(i+2,j),\\ &\forall i,j \;\; p'(i,j)(3,3) = \tilde{p}(i+2,j+2),\\ &\forall i,j \;\; p'(i,j)(1,1) = \tilde{p}(i,j),\\ &\forall i,j \;\; p'(i,j)(1,3) = \tilde{p}(i,j+2). \end{aligned} \tag{P3}$$

The equations (P1), (P2) and (P3) mean that $p' = \theta(p)$, i.e. p' belongs to K. We have $K = K'$ which is hv-local. This completes the proof of the claim.

Now, let us consider a local picture language L over Σ with the set of tiles Δ. We will show that L is the image of a hv-local picture language over $ext(\Sigma)$ by the mapping π. According to (2), $\theta(L)$ is defined on the alphabet Σ_Δ:

$$\Sigma_\Delta = \{a \in ext(\Sigma) \mid T_{2,2}(a) \subseteq \Delta\}$$

and clearly we have:

$$\theta(L) = \{p' \in \Sigma_\Delta^{**} \mid T_{2,1}(p') \cup T_{1,2}(p') \subseteq \Delta'\}.$$

That concludes the proof of Proposition 5 since $L = \pi(\theta(L))$. □

Theorem 7. *Let $L \subseteq \Sigma^{**}$ be a picture language. L is recognizable if and only if there exists a hv-local picture language L' over Σ' and a mapping $\pi : \Sigma' \to \Sigma$ such that $L = \pi(L')$.*

Proof. Let L be a recognizable picture language over Σ. By definition, we know that there exists a local picture language L' over an alphabet Σ' and a mapping $\pi : \Sigma' \to \Sigma$

such that $L=\pi(L')$. According to the Proposition 5, there exists a hv-local picture language L'' over an alphabet Σ'' and $\rho:\Sigma''\to\Sigma'$ such that $L'=\rho(L'')$. So we have $L=\pi(\rho(L''))$ where L'' is hv-local.

Now, let L' be a hv-local picture language over Σ' and $\pi:\Sigma'\to\Sigma$ a mapping. From Proposition 4 it follows that L' is local and so the picture language $L=\pi(L')$ is recognizable. □

Before concluding, we treat the example of a recognizable picture language given by Restivo et al. [2, 3]. This language is the set of all the squares over an alphabet $\Sigma=\{a\}$. We get

$$L=\left\{\begin{array}{|c|}\hline a\\\hline\end{array},\ \begin{array}{|c|c|}\hline a&a\\\hline a&a\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline a&a&a\\\hline a&a&a\\\hline a&a&a\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline a&a&a&a\\\hline a&a&a&a\\\hline a&a&a&a\\\hline a&a&a&a\\\hline\end{array}\cdots\right\}$$

We give a hv-local picture language K associated with L:

$$K=\left\{\begin{array}{|c|}\hline 0\\\hline\end{array},\ \begin{array}{|c|c|}\hline 0&1\\\hline 2&0\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&1&1\\\hline 2&0&1\\\hline 2&2&0\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 2&0&1&1\\\hline 2&2&0&1\\\hline 2&2&2&0\\\hline\end{array}\cdots\right\}$$

We see that $L=\pi(K)$ with $\pi(0)=\pi(1)=\pi(2)=a$. Moreover, K is hv-local for the set of dominoes Δ:

$$\Delta=\left\{\begin{array}{c}\begin{array}{|c|}\hline \#\\\hline \#\\\hline\end{array},\begin{array}{|c|}\hline \#\\\hline 0\\\hline\end{array},\begin{array}{|c|}\hline \#\\\hline 1\\\hline\end{array},\begin{array}{|c|}\hline 0\\\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array},\begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 2\\\hline \#\\\hline\end{array},\begin{array}{|c|}\hline 0\\\hline \#\\\hline\end{array}\\ \begin{array}{|c|c|}\hline \#&\#\\\hline\end{array},\begin{array}{|c|c|}\hline \#&0\\\hline\end{array},\begin{array}{|c|c|}\hline 0&1\\\hline\end{array},\begin{array}{|c|c|}\hline 1&1\\\hline\end{array},\begin{array}{|c|c|}\hline 1&\#\\\hline\end{array}\\ \begin{array}{|c|c|}\hline \#&2\\\hline\end{array},\begin{array}{|c|c|}\hline 2&0\\\hline\end{array},\begin{array}{|c|c|}\hline 2&2\\\hline\end{array},\begin{array}{|c|c|}\hline 0&\#\\\hline\end{array}\end{array}\right\}$$

4. Conclusions

In hv-local picture languages, the horizontal and vertical controls are separated. It is natural to define h-local picture languages and v-local picture languages which are defined by authorized horizontal dominoes in the first case and vertical dominoes in the last one. It is obvious that we cannot obtain all recognizable picture languages by projection of a h-local or a v-local one. This is due, among others, to the fact that we cannot bound the number of rows (resp. columns) in a h-local (resp. v-local) picture language.

It is easy to associate with each h-local (resp. v-local) picture language L a local string language K such that $p\in L$ if and only if each line (resp. row) of p (taken as a string) is in K. Moreover, for each hv-local picture language L with Δ the dominoes set, we have $\Delta=\Gamma_h\cup\Gamma_v$ where Γ_h is a set of horizontal dominoes and Γ_v is a set of vertical dominoes. If we consider the h-local picture language K_h associated with Γ_h and the v-local picture language K_v associated with Γ_v, it is clear that $L = K_h \cap K_v$. So, a recognizable picture language is completely defined by two given local string languages (one for K_h and one for K_v) and a mapping.

Note that for the sake of compactness one can represent a recognizable picture language by two given recognizable string languages and a mapping. For instance the set of all squares over the alphabet $\{a\}$ presented above could be represented by (π, R_h, R_v) with $\pi(0) = \pi(1) = a$ and $R_h = R_v = 0^*10^*$. It is interesting to study the family of picture languages which can be defined by using string languages of the other class of Chomsky hierarchy as done in [4, Section 11].

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